Geophysical Fluid Modeling with the mpi-version of the Princeton Ocean Model (mpiPOM)

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COVER ILLUSTRATION

A three-dimensional surface of near-inertial energy = 0.03 m^2 s^{-2} on Sep/03/12:00, approximately one week after the passage of hurricane Katrina, Aug/25-30/2005, in the Gulf of Mexico, USA, simulated using the Princeton Ocean Model. This shows penetration of intense energies to deep layers due to the presence of the warm-core ring and the Loop Current, both represented by the dark contours in the three cut-away xy-planes. The location of an observational mooring where model-observational analyses were conducted is shown as vertical dashed line [from: Oey et al. 2008: “Stalled inertial currents in a cyclone,” Geophysical Res. Lett., 35, L12604, doi:10.1029/2008GL034273; with permission to reproduce].

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“.. I have no special talents. I am only passionately curious..”
Albert Einstein
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Chapter 1: Basic numerical analysis using finite-differences

The equations that govern the evolution of ocean or atmosphere satisfy the conservation of mass, momentum and tracers such as heat, salt or water vapor, etc. They consist of a set of partial differential equations (PDE’s). For example, the “heat” equation:

\[
\frac{\partial T}{\partial t} = \nabla (\alpha \nabla T), \quad x \in \mathbb{R}_3, \quad t \geq 0
\]  

(1.0-1)

may describe the temperature \( T(x, t) \) (unit: °C or more generally °K) of an enclosed sea contained within the basin \( R_3 \): (longitude \( x \), latitude \( y \) and depth \( z \) (unit: \( m \)) at time \( t \) (unit: \( s \)). Here \( \alpha \) is the diffusivity coefficient (unit: \( m^2 \cdot s^{-1} \)) which may represent the churning or mixing action of the generally turbulent fluid on the temperature of the sea. Fluid temperature can change also due to ocean currents with velocity \( u \) (unit: \( m \cdot s^{-1} \)), so that equation (1.0-1) contains an advective term \( u \cdot \nabla \):

\[
\frac{\partial T}{\partial t} + u \cdot \nabla T = \nabla (\alpha \nabla T), \quad x \in \mathbb{R}_3, \quad t \geq 0.
\]  

(1.0-2)

In this book, boldface indicates a vector. In one-dimensional space, the equation is:

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = \frac{\partial (\alpha \frac{\partial T}{\partial x})}{\partial x}, \quad 0 \leq x \leq L, \quad t \geq 0.
\]  

(1.0-3)

This may describe, for example, the temperature variation of a narrow and straight river of length \( L \). The “\( T \)” may also represent the concentration of some substance (salt), or \( u \) may represent wind and then “\( T \)” the concentration of smoke particles that puff out of a factory chimney, for example.

In numerical models, we approximate the space \( \mathbb{R}_3 \) by dividing it into a finite number of grid boxes; an example for the atmosphere is shown in Fig.1.0-1. On this grid, the governing equations are also approximated, most commonly using “finite-differences.” The approximate governing equations can then be solved on a computer. However, approximations introduce errors and altered physics. In Chapter 1 we study some of the consequences, and learn how to recognize and deal with them.

![Fig.1.0-1 A numerical grid system for the earth’s atmosphere.](image)
Vilhelm Bjerknes [1904] was the first to suggest forecasting the weather using a system of PDE’s, and Lewis Fry Richardson [1922] wrote a book accounting his attempt developing Bjerknes’s techniques. The first success did not come much later however. A paper by Courant, Friedrich and Levy [1928] paved the way by proving that a condition for stability must be met in time-integration of a certain kind of finite-difference equations. John von Neumann developed the first computer (called the ENIAC) at the Institute of Advanced Study in Princeton, where Jule Charney [1950] led a group to conduct the first successful numerical weather prediction.

1.1 One-dimensional finite-difference (FD) approximation

Many basic ideas of numerical analysis can be learned by considering only one-dimensional space. Let “u” be a variable to be modeled; it may represent temperature or velocity:

\[ u = u(x), \quad 0 \leq x \leq L. \quad (1.1-1a) \]

We can use infinite-term Fourier expansion to represent “u”:

\[ u = a_0/2 + \sum_{n=1}^{\infty} (a_n \cos 2\pi n x/L + b_n \sin 2\pi n x/L), \quad n \geq 1. \quad (1.1-2) \]

In this case, all wavelengths are represented: L, L/2, L/3,…, L/\infty.

In numerical models, we approximate “u” by values defined on a finite number of grid points:

\[ u_j = u_j(j\Delta x), \quad j=1,2, \ldots, J+1, \quad \text{so that} \quad \Delta x = L/J. \quad (1.1-1b) \]

With J+1 values of \( u_j \) on the grid, we can only compute a finite number of Fourier coefficients:

\[ a_0, \quad a_1, a_2, \ldots, a_{J/2}, \quad b_1, b_2, \ldots, b_{J/2}. \quad (1.1-3) \]

The component with the shortest wavelength has \( n=J/2 \):

\[ L/(J/2) = 2L/(L/\Delta x) = 2\Delta x. \quad (1.1-4) \]

This means that the shortest scale that the model “can see” (i.e. can resolve) is \( 2\Delta x \). The \( 2\Delta x \)-wavelength is in fact an important scale which if present in a numerical solution usually tells us that the model is not doing a good job approximating the true solution. We will learn about this later.

The derivative \( du/dx \) may be approximated as:

\[ (du/dx)_j \approx (u_{j+1} - u_j)/\Delta x \quad (1.1-5) \]
The RHS is called a finite-difference (FD) approximation of the derivative $du/dx$.

**Accuracy:**

To check how accurate the approximation is, substitute the true solution $u(j\Delta x)$ into the RHS of (1.1-5), and expand using the Taylor series:

\[
\frac{u_{j+1} - u_j}{\Delta x} = \frac{du}{dx}_j + \frac{(d^2u/dx^2)_j \Delta x}{2!} + \frac{(d^3u/dx^3)_j (\Delta x)^2}{3!} + \ldots
\]

Thus the FD-approximation of the derivative consists of the true or exact derivative $(du/dx)_j$ plus some extra terms which are lumped together as:

\[
\varepsilon = O(\Delta x) = \frac{(d^2u/dx^2)_j \Delta x}{2!} + \frac{(d^3u/dx^3)_j (\Delta x)^2}{3!} + \ldots
\]  

(1.1-6)

We then say that the order of accuracy of the finite-difference approximation (1.1-5) is “$\Delta x$,” or $O(\Delta x)$ (read as Order of $\Delta x$). The FD approximation (1.1-5) is called a first order approximation. Notice that the idea of “order of accuracy” may not be meaningful in regions where, for a fixed grid size $\Delta x$, the solution changes rapidly (with $x$) – say when $u$ has strong gradients with rapidly changing inflexion points, so that $d^2u/dx^2$ and higher derivatives can be large. In that situation, the idea of “$O(\Delta x)$” is useful only if $\Delta x$ is made finer.

**Consistency:**

To be consistent, the FD approximation of the derivative (e.g. 1.1-5) must approach the true derivative, i.e. $\varepsilon \sim 0$ as $\Delta x \to 0$. Clearly, (1.1-5) is a consistent approximation to $du/dx$.

**Finite difference schemes:**

The algebraic equation obtained when derivatives in a differential equation are replaced by FD approximation is called a finite difference approximation to that differential equation, or a finite difference scheme.

Linear advection of the variable “$u$” by a velocity “$c$” is described by:

\[
\partial u/\partial t + c \partial u/\partial x = 0, \quad u = u(x,t), \quad c = \text{positive constant.}
\]  

(1.1-7)

The general solution is:

\[
u = f(x-ct), \quad \text{where } f = \text{arbitrary function.}
\]  

(1.1-8)
The “f” is determined by initial condition. Thus, if
\[ u(x,0) = F(x), \] then \[ u = F(x - ct) \] is the solution. \hspace{1cm} (1.1-9)

**Fig.1.1-1** One of the characteristics \( x - ct = \text{constant} \) of (1.1-7).

As time marches forward, the true solution depends on its value “to the left” or “upstream.” This suggests the following FD scheme:

\[ \frac{u_j^{n+1} - u_j^n}{\Delta t} + c \frac{u_j^n - u_{j-1}^n}{\Delta x} = 0 \] \hspace{1cm} (1.1-10)

**Forward Difference** \hspace{1cm} **Upstream Difference**

The truncation error is obtained by substituting the true “u”, expand in Taylor series about \( j\Delta x \) and \( n\Delta t \), and then subtracting \( (\partial u/\partial t)_j^n + c(\partial u/\partial x)_j^n \) at \( j\Delta x \) and \( n\Delta t \) (denoted by \( (\cdot)_j^n \)):

\[
\varepsilon = [u (j\Delta x, (n+1)\Delta t) - u (j\Delta x, n\Delta t)]/\Delta t + c [u (j\Delta x, n\Delta t) - u ((j-1)\Delta x, n\Delta t)]/\Delta x
- (\partial u/\partial t)_j^n - c (\partial u/\partial x)_j^n
\]

\[
= [(\partial^2 u/\partial t^2)\Delta t/2 + (\partial^3 u/\partial t^3)\Delta t^2/3! + \ldots] + c [(\partial^2 u/\partial x^2)\Delta x/2 + (\partial^3 u/\partial x^3)\Delta x^2/3! + \ldots]
\]

\[
= O(\Delta x, \Delta t). \hspace{1cm} (1.1-11)
\]

The FD-scheme (1.1-10) is an *explicit* scheme, so called because the FD-solution at the forward time \( “n+1” \), and any \( j\Delta x \), i.e. \( u_j^{n+1} \), can be explicitly calculated once the \( u \)-values at the previous time \( “n” \) are known.

**Consistency of the FD scheme:**

As \( \Delta x \) and \( \Delta t \to 0 \), the above FD scheme (1.1-10) is said to be *consistent* since it then becomes closer and closer to the actual differential equation (1.1-7).

**Convergence:**

A FD solution is said to be *convergent* if for a fixed total time, \( u_j^n \to u(j\Delta x, n\Delta t) \) as \( \Delta x \) and \( \Delta t \to 0 \).
A FD scheme is said to be convergent if it gives a convergent solution for any initial conditions.

A consistent FD scheme does not necessarily mean that its solution approaches the true solution (see Fig.1.1-2).

**Fig.1.1-2** An example of a consistent FD scheme (1.1-10) which does not yield a convergent solution.

In Fig.1.1-2, solid line is the true-solution characteristic passing through the origin (0,0) and the square grid point □ (“A”) where/when the approximate FD solution (∼ u(0,0)) is desired. The slope of the characteristic is \( \frac{dt}{dx} = \frac{1}{c} \). On the other hand, using (1.1-10), the FD-solution at □ depends only on those circle ○ grid points. The region defined by the circle ○ grid points is called the domain of dependence of the FD scheme. As we refine \( \Delta x \) and \( \Delta t \) keeping their ratio the same as that shown by the grid rectangles, the FD-solution at □ may be arbitrarily different from \( u(0,0) \), because values at ○ may be very different from the \( u(0,0) \) value. For the FD solution to “know” the \( u(0,0) \) value, it is clear that the FD’s domain of dependence must include the origin, i.e. the ratio \( \frac{\Delta t}{\Delta x} \) must be chosen such that it is less than the slope of the solution characteristic “1/c”:

\[
\frac{\Delta t}{\Delta x} \leq \frac{1}{c}, \quad \text{or} \quad c \frac{\Delta t}{\Delta x} \leq 1 \quad \text{(1.1-12)}
\]

In Fig.1.1-2, this can be accomplished by halving the \( \Delta t \) while keeping the \( \Delta x \) the same. We note that for the special case when \( \Delta t \) and \( \Delta x \) are chosen such that \( c \frac{\Delta t}{\Delta x} = 1 \), the FD-solution at □ will be *exactly* equal to the true solution = \( u(0,0) \). This is also seen from (1.1-10) which then gives \( u_{i}^{n+1} = u_{i-1}^{n} = u_{i-2}^{n-1} = \ldots \) etc.

Condition (1.1-12) is a necessary condition for convergence of the FD scheme (1.1-10). The condition is necessary because if we violate the condition, then we know for sure that the solution will not converge. In other words, making sure that the domain of dependence of the FD includes the solution characteristic cannot guarantee that the FD-solution converges. Something else may also need to be satisfied, as we will now learn.

**Stability:**

What is the behavior of the FD-solution as time-stepping proceed?
Assuming that the true solution is bounded, then a FD-solution $u_j^n$ is stable if its error $|u_j^n - u(j\Delta x,n\Delta t)|$ remains bounded as “n” increases, for fixed values of $\Delta t$ and $\Delta x$.

A FD-scheme is stable if its solution is stable for any initial conditions.

To guarantee that a FD-scheme is stable, we will derive the condition that ensures that the maximum absolute magnitude of the FD solution at a particular time-level “n+1” (say) is less than the corresponding maximum absolute magnitude at the previous time step “n.” We test to make sure that the scheme is stable by considering all possible perturbations. This condition will then be a sufficient condition, but obviously it may not be necessary.

We again take the FD-scheme (1.1-10) as an example, and find the sufficient stability condition. Eqn. (1.1-10) can be written as:

$$u_j^{n+1} = (1-\mu) u_j^n + \mu u_{j-1}^n,$$

$\mu = c\Delta t/\Delta x > 0$ (1.1-13)

*Direct method:*

If the coefficients of $u_j^n$ and $u_{j-1}^n$ are positive, i.e. if

$$\mu = c\Delta t/\Delta x \leq 1, \text{ and } \mu > 0$$

then

$$|(1-\mu) u_j^n + \mu u_{j-1}^n| \leq (1-\mu) |u_j^n| + \mu |u_{j-1}^n|.$$ 

Therefore, taking absolute maximum over all j’s of both sides of (1.1-13):

$$\text{Max}_j |u_j^{n+1}| \leq (1-\mu) \text{Max}_j |u_j^n| + \mu \text{Max}_j |u_{j-1}^n| \leq \text{Max}_j |u_j^n|$$

which proves the solution remains bounded with time-stepping. The condition (1.1-14) is therefore a sufficient condition for stability of the FD scheme (1.1-10). Note that this same condition *happens to be also* the necessary condition for convergence.

*Energy method:*

By squaring both sides of (1.1-13) and summing over all j’s, the same condition (1.1-14), coupled with x-periodic boundary condition, can be shown to also lead to (after some algebra):

$$\Sigma_j(u_j^{n+1})^2 \leq \Sigma_j(u_j^n)^2$$

i.e. each and every value of $u_j^n$ must also be bounded with time-stepping.
von Neumann (Fourier) method:

This method borrows the idea from the analytical method of applying a single Fourier harmonic mode to (1.1-7) by the separation of variables:

\[ u(x,t) = \text{Re} \{ U(t) e^{ikx} \} \]  

so that (1.1-7) becomes:

\[ \frac{dU}{dt} = -ikcU \rightarrow U(t) = U(0) e^{-ikct} \quad \text{and the solution is then:} \]

\[ u(x,t) = \text{Re} \{ U(0) e^{ik(x-ct)} \} \]  

This shows that each harmonic component (with wavenumber k) is advected at the constant speed “c” with unchanging amplitude.

For the FD-scheme (1.1-13), we apply the same idea and assume a solution of the form:

\[ u^n_j = \text{Re} \{ U^{(n)} e^{ikj\Delta x} \} \]  

where \( U^{(n)} \) is the amplitude of the FD-solution at time level “n.” Substitute into (1.1-13):

\[ \frac{U^{(n+1)}}{U^{(n)}} \equiv \lambda = (1 - \mu) + \mu e^{ik\Delta x} \]  

where \( |\lambda| \) is the amplification factor. Therefore, for the solution to be bounded with time-stepping, we require that \( |\lambda| \leq 1 \). Evaluating \( |\lambda| \):

\[ |\lambda| = 1 + 2\mu (1 - \mu) [-1 + \cos(k\Delta x)] \]  

It is clear that for \( |\lambda| \) to be \( \leq 1 \):

\[ 2\mu (1 - \mu) [-1 + \cos(k\Delta x)] \leq 0 \]

which can only be satisfied for all k’s if:

\[ \mu = c\Delta t/\Delta x \leq 1 \]  

which is again the same stability condition (1.1-14). This condition is commonly known as the CFL or Courant-Friedrich-Levy stability condition, named in honor of the authors who first derived it.
**An implicit scheme for the diffusion equation:**

The CFL condition can place an extremely strict (i.e. very small) upper limit for the size of the time step $\Delta t$ especially when the spatial grid size (e.g. $\Delta x$) is very small. This is the case in the ocean and atmosphere for the vertical direction because the layer is very thin (~10km) compared to horizontal distances ~ 1000 km. So when finite-differencing in the z-direction, the $\Delta z$ can be very small. Implicit scheme removes the restriction on $\Delta t$ imposed by the smallness of $\Delta x$.

Consider \[ \frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \] (1.1-20)
and we approximate it using the following implicit scheme:

\[ \frac{u_j^{n+1} - u_j^n}{\Delta t} = \alpha \frac{(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1})}{\Delta x^2}. \] (1.1-21)

In contrast to the explicit scheme (1.1-10), the FD-solution at the forward time “n+1”, and at any $j\Delta x$, i.e. $u_j^{n+1}$, can *not* now be calculated unless we also know the values at the neighboring “j-1” and “j+1” locations, because at these locations the “u” is also defined at the forward time “n+1” – the $u_{j+1}^{n+1}$ and $u_{j-1}^{n+1}$ on the RHS of above equation. Therefore the $u_j^{n+1}$ for all j’s must be solved simultaneously. Such a scheme is called an implicit scheme.

Applying the von Neumann analysis, we get:

\[ |\lambda_{imp}| = 1/[1 + 2\gamma (1 - \cos(k\Delta x))], \quad \text{where } \gamma = \alpha \Delta t/\Delta x^2 \]

Clearly, $|\lambda_{imp}| \leq 1$ always regardless of the values of $\Delta t$ and/or $\Delta x$. For this reason, implicit scheme is used in ocean and atmospheric models especially when approximating the z-direction.

**Von Neumann Analysis Applied to the Leap-Frog Scheme:**

\[ (u_j^{n+1} - u_j^{n-1})/2\Delta t + c (u_{j+1}^n - u_{j-1}^n)/2\Delta x = 0 \] (1.1-22)

Rearrange,

\[ u_j^{n+1} = u_j^{n-1} + (c\Delta t/\Delta x) (u_{j+1}^n - u_{j-1}^n) \] (1.1-23a)

\[ u_j^n = u_j^n \] (1.1-23b)

where we have added a supplementary (trivial) equation (1.1-23b) the reason will be clear below.

Then after substituting (1.1-17):
\[ U^{(n+1)} = -2i\mu \sin(k\Delta x), \quad U^{(n)} + U^{(n-1)} \]  
**Equation 1.1-24a**

\[ U^{(n)} = U^{(n)} \]  
**Equation 1.1-24b**

In vector/matrix form:

\[ V^{(n+1)} = A.V^{(n)} \]  
**Equation 1.1-25**

where \( V^{(n+1)} = (U^{(n+1)}, U^{(n)})^T, \quad V^{(n)} = (U^{(n)}, U^{(n-1)})^T, \) and

\[ A = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{where} \quad a = -2i\mu \sin(k\Delta x) \]  
**Equation 1.1-26**

Compare (1.1-25) with (1.1-18) and we see that the matrix \( A \) takes the role of the amplification factor \( \lambda \). However, we now have a matrix, and it is known in Linear Algebra that the “size” of a matrix is measured by its eigenvalue for which we will use the same symbol “\( \lambda \).” To find the eigenvalue of \( A \), we solve for

\[ |A - \lambda I| = 0 \]  
**Equation 1.1-27**

where \( |.| \) is the determinant, and \( I \) is the identity matrix:

\[ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]  
**Equation 1.1-28**

In the case of (1.1-18), we may rewrite it in the same form as (1.1-25): \( U^{(n+1)} = A.U^{(n)} \), and its corresponding (1.1-27) is then \( A - \lambda I = 0 \), or simply \( A = \lambda I \) which we studied before (see descriptions after eqn.1.1-18) to determine the stability condition of the forward-time-upstream scheme (1.1-10).

In the present case, (1.1-27) becomes:

\[ \begin{vmatrix} a - \lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0, \quad \text{i.e.} \quad \lambda^2 - a\lambda - 1 = 0 \]  
**Equation 1.1-29**

whose solution is:

\[ \lambda = \{ a \pm [a^2 + 4]^{1/2} \}/2 \]

or,

\[ \lambda = -i\mu \sin(k\Delta x) \pm [\mu^2 \sin^2(k\Delta x)+1]^{1/2} \]  
**Equation 1.1-30**

If \( \mu^2 \sin^2(k\Delta x) > 1 \), then \( [..]^{1/2} \) is imaginary = \( i [\mu^2 \sin^2(k\Delta x) - 1]^{1/2} \) and therefore:

\[ \lambda = i \{-\mu \sin(k\Delta x) \pm [\mu^2 \sin^2(k\Delta x)-1]^{1/2} \}, \quad \mu \sin(k\Delta x) > 1 \]  
**Equation 1.1-31**
Clearly, one of the 2 roots of (1.1-31) has $|\lambda| > 1$ (the minus case), indicating instability.

If $\mu^2 \sin^2(k\Delta x) < 1$, then, for both roots, we have:

$$|\lambda_{\pm}|^2 = \mu^2 \sin^2(k\Delta x) + [-\mu^2 \sin^2(k\Delta x) + 1] = 1.$$ 

So the stability condition is:

$$\mu = c\Delta t/\Delta x \leq 1$$  \hspace{1cm} (1.1-32)

which is the same CFL condition as the forward-time-upstream scheme (see (1.1-14)).

**Boundary conditions:**

Closed (e.g. at the coast) and ocean’s bottom and surface boundary conditions are fairly straightforward. The trickiest boundaries to treat are “open” boundaries in regional modeling. The best way to learn is by examples which we will do in later chapters when experimenting with POM. I have also written a set of notes for their treatments, and you may download them from:


Another way to do open-boundary conditions is by doing 2-way nesting; see this report:


**Homework 1:**

1. P. 122 Derive (1.1-11) to 2nd order accuracy.

2. P. 126 Derive (1.1-19)

3. Derive “Stability condition of FTCS”

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = c \left( \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \right)$$

4. Derive “FTCS for diffusion”:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \alpha \left( \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \right)$$

then solve for $u$ in $0 \leq x \leq 100$ km and for $0 \leq t \leq 100$ days using $\alpha = 10$ m$^2$/s. The initial and boundary conditions are: $u(x, t=0) = 0$, $u(x=0, t) = 0$ & $u(x=100$ km, $t)=1$. Prepare a *ppt* to discuss the results.
5. Formulate the matrix for diffusion equation above but for an implicit scheme (RHS all at \( n+1 \) time step) & then solve (e.g. using Matlab) and compare with the solution in “4”. Prepare a *ppt* to discuss the results.

*6. Solve using (a) “3” and “4”, and then (b) “3” and “5” the following advection-diffusion equation (with \( c = 0.2 \text{ m/s} \)):

\[
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}
\]

Prepare a *ppt* to discuss the results.

Solution to Homework 1.6

In this homework, you will learn how to apply von Neuman analysis to the advection-diffusion equation (H.1.6-1) which commonly occurs in equations governing fluid flows. We apply FTCS: Forward Time Centered Space scheme which was one of the early schemes applied in oceanic and atmospheric models, and is still being used today. We will learn that oscillations can occur under certain conditions and that these oscillations may not be instability.

Stability analysis:

Von Neuman analysis gives \( U^{(n+1)}/U^{(n)} = \lambda \) as:

\[
\lambda = 1 - 2\gamma (1 - \cos \theta) - i\mu \sin \theta, \text{ where } \mu = c\Delta t/\Delta x, \gamma = \alpha \Delta t/\Delta x^2 \text{ & } \theta = k\Delta x
\]  

(H.1.6-2a)

Write the complex \( \lambda = \lambda_r + i\lambda_i \), then

\[
[\lambda_r - (1 - 2\gamma)] + i\lambda_i = 2\gamma \cos \theta - i\mu \sin \theta
\]  

(H.1.6-2b)

We may compare this with the equation of an ellipse centered at \((x_o, y_o)\) (Fig.H.1.6-1):

\[
x - x_o = 2b \cos \theta \text{ and } y - y_o = a \sin \theta, \text{ where } x = \lambda_r, y = \lambda_i, x_o = 1 - 2\gamma, y_o = 0, b = \gamma \text{ & } a = \mu
\]

In Fig.H.1.6-1, the red ellipse is centered at \((1 - 2b, 0)\); it represents all points traced by \( \lambda \) as \( \theta \) varies through all possible values from 0 to \( 2\pi \). For stability, the ellipse must be wholly enclosed by the unit (blue) circle. For this to be true, it is necessary that:

\[
2b < 1, \text{ i.e. } \gamma = \alpha \Delta t/\Delta x^2 < \frac{1}{2}, \text{ i.e. } \Delta t < \Delta x^2/(2\alpha),
\]  

(H.1.6-3a)

which is the stability condition for FTCS for the diffusion equation.
Fig. H.1.6-1 Illustrating stability of FTCS for advection-diffusion equation: the “a” and “b” must be chosen so that the red ellipse is entirely inside the unit (blue) circle.

From Fig. H.1.6-1, it is also necessary that $a < 1$ and $a < 2b$; the second of these gives:

$$c\Delta t/\Delta x < 2\alpha\Delta t/\Delta x^2, \quad \text{or} \quad R_c = c\Delta x/\alpha < 2 \quad (H.1.6-3b)$$

The $R_c$ is called the cell Reynolds number. The above 2 conditions can be combined to obtain alternate formulae. From the first one:

$$2\Delta t/\Delta x < \Delta x/\alpha < 2/c \quad (from \ the \ second \ one) \Rightarrow \quad c\Delta t/\Delta x = \mu < 1. \quad (H.1.6-4a)$$

Also, squaring the second one:

$$4\alpha/c^2 > \Delta x^2/\alpha > 2\Delta t \quad (from \ the \ first \ one) \Rightarrow \quad \Delta t < 2\alpha/c^2. \quad (H.1.6-4b)$$

An example of an unstable solution, with $\Delta x$, $\Delta t$, $\alpha$ & $c$ chosen to violate (H.1.6-3a,b) or (H.1.6-4a,b) can be seen in a number of your homework solutions using $c=0.2 \text{ m/s}$, $\alpha=10 \text{ m}^2/\text{s}$:

(a) Wolf’s [http://oeylectures.pbworks.com/w/file/69702907/HW_A6_a.ppt](http://oeylectures.pbworks.com/w/file/69702907/HW_A6_a.ppt), slide#2: $\Delta t=7200 \text{s}$ & $\Delta x=2000 \text{m}$. Thus $\alpha\Delta t/\Delta x^2=10\times7200/(4\times10^6)<1/2$ but $R_c=(0.2\times2000/10)>2 \quad \text{violating (H.1.6-3a,b)}$; OR: $c\Delta t/\Delta x=0.2\times7200/2000<1$ but $\Delta t/\Delta x=7200\times0.2/20>1 \quad \Rightarrow \text{violating (H.1.6-4a,b).}$

Wolf tried FTFS (slides#3&4) and finds that the solution is stable;

(b) Roger’s [http://oeylectures.pbworks.com/w/file/70471646/roger_HWA6.ppt](http://oeylectures.pbworks.com/w/file/70471646/roger_HWA6.ppt), slide#3: $\Delta t=3600 \text{s}$ & $\Delta x=500 \text{m}$. Thus $\alpha\Delta t/\Delta x^2=10\times3600/(2.5\times10^3)<1/2$ but $R_c=(0.2\times500/10)>2 \quad \text{violating (H.1.6-3a,b)}$; OR: $c\Delta t/\Delta x=0.2\times3600/500>1$ AND $\Delta t/\Delta x=3600\times0.2/20>1 \quad \Rightarrow \text{violating (H.1.6-4a,b).}$
Roger also completed the solution using implicit scheme for both the $c \partial u / \partial x$ & $\alpha \partial^2 u / \partial x^2$ terms (i.e. using “n+1” for these terms). He formulated and solved the matrix equations and the solution is STABLE as seen in his slide#5 – some of his plots are reproduced below. These show stable but oscillatory steady-state solution after a “long time” ($t > 2$ days).

![Figure H.1.6-2 RG’s solution at various indicated times showing oscillations which are the correct solution of the finite-difference equation: $R_c = 10$.](image)

(c) Yoyo also did this HW: [http://oeylectures.pbworks.com/w/file/70386197/20131029_Yoyo_HWA6.pptx](http://oeylectures.pbworks.com/w/file/70386197/20131029_Yoyo_HWA6.pptx) which shows unstable solution (slide#1) and oscillatory solution (slide#2); Fig.H.1.6-3.

![Figure H.1.6-3 YY’s solution showing oscillations which are also the correct solution of the finite-difference equation: $R_c = (0.2 \times 2000/10) = 40$.](image)
(d) Chichien (http://oeylectures.pbworks.com/w/file/70453356/A6.pptx) was able to stabilize the FTCS because he chose $\Delta t=0.05$ & $\Delta x=1$; Thus $\alpha \Delta t/\Delta x^2=10\times0.05/1<1/2$ & $R_c=(0.2\times1/10)<2 \rightarrow$ satisfying (H.1.6-3a,b); OR: $c\Delta t/\Delta x=0.2\times0.05/1<1$ & $\Delta t c^2/(2\alpha)=0.05\times0.2^2/20<1 \rightarrow$ satisfying (H.1.6-4a,b).

![FTCS for both time and space](image)

Fig.H.1.6-4 CC’s solution showing no oscillations: $R_c = 0.02$ (red) and $R_c = 0.04$ (blue).

(e) Two sets of good calculations are given by Tank (K.J. Lin; http://oeylectures.pbworks.com/w/file/70692496/Homework%20A6.pptx). In the first set, he chose the parameters to satisfy $R_c < 2$ but one calculation violates the diffusion stability condition (H.1.6-3a) and the other satisfies it – these 2 calculations are shown in Fig.H.1.6-5.

![Diffusion equation–FTCS scheme](image)

Fig.H.1.6-5 Tank’s solutions both with FTCS but left (unstable) is for $\alpha=10\text{m}^2/\text{s}$, $c=0.2\text{m/s}$, $\Delta t=0.5\text{s}$ and $\Delta x=2\text{m}$, and right (stable) is same but $\Delta x=4\text{m}$. Both has $R_c < 2$. 


In the 2nd set (Fig.H.1.6-6), Tank chose \( R_c > 2 \), and used FTCS and implicit schemes, figure below:

Fig.H.1.6-6 Tank’s solutions for \( R_c = 20 \) (\( \alpha = 10 \text{m}^2/\text{s} \), \( c = 0.2 \text{m/s} \), \( \Delta t = 1800 \text{s} \) and \( \Delta x = 1 \text{km} \)), using left: FTCS which displays oscillatory *unstable* solution, and right: implicit CS scheme which displays an oscillatory but stable solution.

The above solutions in Fig.H.1.6-2-6 are correct steady-state finite-difference (FD) solution to

\[
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}
\]

using the centered space differencing with different \( R_c \). The oscillations when \( R_c \) is large are *not* due to instability. The oscillations arise because of the inability for the finite-difference grid to “see” the solution once \( R_c \) exceeds 2. To show this, it is convenient to use the non-dimensionaized form of the equation:

\[
R_e \frac{\partial u}{\partial x} = \frac{c}{L} \frac{\partial^2 u}{\partial x^2}, \quad R_e = \frac{cL}{\alpha}, \quad u(0) = 0 \text{ & } u(1) = 1,
\]  

(H.1.6-5)

where \( x \) has been nondimensionalized by the domain length \( L \), so that \( 0 \leq x \leq 1 \). Using FTCS:

\[
(R_c - 2)u_{j+1} + 4u_j - (R_c + 2)u_{j-1} = 0, \quad R_c = \frac{c\Delta x}{\alpha}, \quad 2 \leq j \leq jm - 1, \quad u_1 = 0 \text{ & } u_{jm} = 1;
\]  

(H.1.6-6)

see Fig.H.1.6-5. Dividing by \( u_{j+1} \) assumed non-zero:

\[
(R_c - 2) + 4G_j - (R_c + 2)G_{j-1} = 0, \quad G_j = \frac{u_j}{u_{j+1}} \text{ & } G_{j-1} = \frac{u_{j-1}}{u_{j+1}}
\]  

(H.1.6-7)

Note: \(|G_{j-1}| < |G_j| < 1\).  

(H.1.6-8)
For $R_c < 2$: we guess by inspection of (H.1.6-7) that $G_j$ and $G_{j-1}$ are positive. Assuming this, that equation becomes:

$$(R_c - 2) + 4G_j = (R_c + 2)G_{j-1} < (R_c + 2)G_j,$$  
using (H.1.6-8).  

Therefore

$$(2 - R_c)G_j < (2 - R_c)G_{j-1}.$$  

If $G_j$ were negative, then the above inequality (H.1.6-9) becomes instead:

$$(R_c - 2) - 4|G_j| < -(R_c + 2)|G_j|,$$  

$$
\Rightarrow (R_c - 2)(1 + |G_j|) > 0
\Rightarrow (1 + |G_j|) < 0 \text{ (since } R_c - 2 < 0),
$$

which is impossible. Therefore, for $R_c < 2$, $G_j$ and $G_{j-1}$ are positive and bounded by 1. The solution is smooth, of one sign (Fig.H.1.6-5a).

For $R_c = 2$: (H.1.6-7) becomes: $G_j = G_{j-1}$, or $u_j = u_{j-1}$. Applying the boundary condition at $x = 0$, $u_1 = 0$, we then have $u_j = 0$ for all $j < jm$, except at $jm$ where the boundary condition is $u_{jm} = 1$ (Fig.H.1.6-5a).

We see then that as the wind (i.e. $c$) increases, $R_c$ increases, and the “$u$” profile is blown to the right until at $R_c = 2$ (Fig.H.1.6-5a) when the foot of the profile is next to the last grid point ($j = jm - 1$), and the “$u$” then jumps from 0 to 1 across only 1 grid cell. For stronger wind, $R_c > 2$, we can anticipate that the FD-scheme cannot handle because the foot is now blown past $j = jm - 1$ which the numerical solution cannot “see.”

For $R_c > 2$: We assume again that $G_j$ and $G_{j-1}$ are positive. Inequality (H.1.6-9) becomes:

$$(R_c - 2)(1 + G_j) < 0,$$

which implies that $(1 + G_j) < 0$, contradicting the assumed positive $G_j$. Therefore, $G_j$ must be negative,

$$G_j < 0,$$  

leading to a consistent (H.1.6-10) that $(1 + |G_j|)$ must be $> 0$ since now $(R_c - 2) > 0$.

Let $j = jm - 1$, then applying (H.1.6-11):
Then let \( j = jm - 2 \), so that \( G_{jm-2} = \frac{u_{jm-2}}{u_{jm-1}} < 0 \); therefore \( u_{jm-2} > 0 \). Continue this way, working backward, \( j = jm - 3, jm - 4, \ldots, 3, 2 \), we get oscillatory solution – see Fig.H.1.6-5b.

**Summary:**

- For time-integration using FTCS applied to \( \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2} \), stability requires (i) a diffusion-type condition \( \Delta t < \Delta x^2/(2\alpha) \) and (ii) a limitation on the cell Reynolds number \( R_c = c\Delta x/\alpha < 2 \). If either of these is violated, the solution grows rapidly and can display oscillations.

- Centered-space differencing applied to the steady equation: \( c \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2} \), which may be obtained by either directly solving it using matrix, or time-stepping using an implicit scheme, displays oscillations when \( R_c > 2 \), but this has nothing to do with instability. The oscillations are in fact the exact solution of the FD scheme: its “response” to the fact that it can no longer see or resolve the fast changing solution.

The last point in the above summary is further understood by examining the exact solution to (H.1.6-5), which is a Home Work problem #7 below.

7*. Derive and thoroughly discuss the analytical solution to (H.1.6-5):

\[
R_c \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}, \quad u(0) = 0 \quad \text{&} \quad u(1) = 1,
\]  

(H.1.6-5)

Pay particular attention to the character of the solution as \( R_c \) becomes large. Discuss the solution in terms of boundary layer. Then discuss the solution in view of what you have learned above using centered-space FD method.

8*. Suppose instead of centered space, we use one-sided (or upwind) differencing for \( \partial u / \partial x \approx (u_j - u_{j+1})/\Delta x \). Compute the numerical solution and thoroughly discuss it (e.g. do oscillations occur, why or why not? etc.).
Chapter 2: Homogeneous flow with sloping topography - standing wave

In this chapter we run the experiment “exp001” of mpiPOM: (ftp://profs.princeton.edu/leo/mpipom/atop/tests/exp001/). This experiment encompasses various basic ideas of geophysical fluid dynamics. We first describe the set-up, physics and goal of the experiment. Next we derive the necessary equations and explain the theory. We then describe in general what we can anticipate from the model solution. After running the experiment, we analyze the results to check against the theory, discuss discrepancies between theory and experiment, and finally, analyze the results to understand the discrepancies.

Fig.2.1-1 mpiPOM exp001 domain: topographic standing wave on an f-plane channel. Scale is approximate: eastern portion is actually longer ~2000 km than western portion ~ 600 km. Study region (~ 330km×200km) with ridge is gray-shaded, and has a wall along its northern edge shown as the thin line. Arrows show schematically flow directions. Grid sizes Δx = Δy = 11.12 km.

2.1 Set-up, physics and goal of mpiPOM exp001

The domain is a rotating (ocean) channel which is closed on all 4 sides (walls) with west-east length \( L_x \approx 2500 \text{ km} \) and south-north \( L_y \approx 400 \text{ km} \) as shown in Fig.2.1-1. The Coriolis parameter “f” is constant = \( 6\times10^{-5} \text{ s}^{-1} \) corresponding to 24.3oN latitude. Water depth “H” is deeper in the south where \( H \approx 85 \text{ m} \) and it decreases linearly towards the northern coast where \( H \approx 10 \text{ m} \), except in the shaded “study” region where there is a ridge (“seamount”). This set-up was to study standing-wave circulation in the Taiwan Strait where there is a ridge jutting out of Taiwan in the mid-strait over which a branch of the Kuroshio flows [Oey et al, 2013]. In the model, the incoming background flow is eastward past the ridge, instead of northward as in the real Kuroshio. The essential dynamics is the same since \( f = \text{constant} \) and with additional assumptions, as we shall see, topography exerts an important control. The region to the right of the study region may be thought of as the sloping shelf of the East China Sea, though in the experiment it is treated as simply a reservoir of water which supplies flow into the Taiwan Strait. To generate the background flow past the ridge, a spatially constant, steady \textit{westward} wind stress \( \tau^w = \tau^w_{ox} = -0.7\times10^{-4} \text{ m}^2 \text{ s}^{-2} \) is imposed; note: the \( \tau^w \) is kinematic, i.e. it is = wind stress in N m\(^{-2}\) divided by sea-water density \( \rho_o \approx 1025 \text{ kg m}^{-3} \). The goal is to simulate and explain the resulting circulation as the background flow is perturbed by the ridge.
2.2 The governing equations and assumptions

Assuming that sea-water is incompressible, the following continuity and momentum (fixed to the rotating earth, written in vector-invariant form) equations apply [see Gill, 1982]:

\[ \nabla \cdot \mathbf{u} = 0 \]  \hspace{1cm} (2.2-1a)

\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{f}_a \times \mathbf{u} = -\nabla (g z + p/\rho + |\mathbf{u}|^2/2) + \partial (K_M \partial \mathbf{u}/\partial z)/\partial z \]  \hspace{1cm} (2.2-1b)

where \( \nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \), \( \mathbf{u} = (u,v,w) \), \( \mathbf{f}_a = 2\Omega + \mathbf{\xi} \) is the absolute vorticity, \( \Omega = \) earth’s rotation vector, \( \mathbf{\xi} = \nabla \times \mathbf{u} \) is the relative vorticity vector, \( g \) is acceleration due to gravity, \( p \) is pressure, \( \rho \) is density, \( K_M \) is kinematic eddy viscosity (unit: \( m^2 \, s^{-1} \)), and we have used:

\[ D\mathbf{u}/Dt = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{\xi} \times \mathbf{u} + \nabla(|\mathbf{u}|^2/2), \]  \hspace{1cm} (2.2-2)

to rewrite the material derivative \( D/Dt \) in (2.2-1b) in vector-invariant form (appears the same in all coordinate systems). Let \( \mathbf{x} \) be oriented such that the \( z \)-direction is in the same direction as \( \Omega \), so:

\[ \mathbf{f}_a = (\mathbf{\xi}_a, \mathbf{\xi}_a + f+\mathbf{\zeta}) \]  \hspace{1cm} (2.2-3)

where \( \mathbf{\zeta} = \mathbf{k} \nabla \times \mathbf{u} \) is the z-component of the relative vorticity and \( f = 2|\mathbf{\Omega}| \sin(\theta), \theta = \) latitude. For large-scale, nearly-horizontal flows, \( \mathbf{f}_a \approx (0, 0, f+\mathbf{\zeta}) = \mathbf{k} (f+\mathbf{\zeta}) \) but this needs not be assumed. Also, the vertical dimension is much smaller than horizontal, \( H \ll L_x \) (or \( L_y \)), and the flow is then approximately hydrostatic, so that the \( z \)-component of (2.2-1b) becomes:

\[ \frac{\partial p}{\partial z} = -\rho g; \quad \text{or} \quad \frac{\partial p'}{\partial z} = -\rho' g \quad \text{and} \quad \frac{dp}{dz} = -\rho_o g, \]  \hspace{1cm} (2.2-4)

where the primes denote perturbations, and we have, for convenience, split the \( p \) and \( \rho \) as:

\[ p = p_o(z) + p'(x,t) \quad \text{and} \quad \rho = \rho_o(z) + \rho'(x,t), \]  \hspace{1cm} (2.2-5)

i.e. as the sum of a vertical profile plus deviation. In the ocean \( |\rho'| \ll \rho_o \); then, the \( p/\rho \) in the \( x \) and \( y \)-component of the momentum equation (2.2-1b) may be replaced by \( p/\rho_o \approx p/\rho_o \) where \( \rho_o = \) constant (Bousinnesq assumption):

\[ \frac{\partial \mathbf{u}_h}{\partial t} + \mathbf{k} (f+\mathbf{\zeta}) \times \mathbf{u}_h + w \partial \mathbf{u}_h/\partial z = -\nabla_h (p/\rho_o + |\mathbf{u}_h|^2/2) + \partial (K_M \partial \mathbf{u}_h/\partial z)/\partial z \]  \hspace{1cm} (2.2-6a)

where \( \mathbf{u}_h = (u,v) \) and \( \nabla_h = (\partial/\partial x, \partial/\partial y) \) are the horizontal velocity and gradient operator (i.e. differentiation keep \( z \) constant), and for convenience the prime on \( p \) is omitted. In terms of \( D/Dt \):

\[ \frac{\partial \mathbf{u}_h}{\partial t} + \mathbf{u} \nabla \mathbf{u}_h + \mathbf{k} \times \mathbf{u}_h = -\nabla_h p/\rho_o + \partial (K_M \partial \mathbf{u}_h/\partial z)/\partial z \]  \hspace{1cm} (2.2-6b)
2.3 The depth-averaged equations

Integrate \( \nabla \cdot \mathbf{u} = 0 \) across the water depth from \( z_B \) to \( z_T \) and integrate by parts:

\[
\int_{z_B}^{z_T} \nabla \cdot \mathbf{u} \, dz = \nabla h \cdot \mathbf{U} - \mathbf{u}^h_T \cdot \nabla h z_T + \mathbf{u}^h_B \cdot \nabla h z_B + w_T - w_B = 0 \tag{2.3-1a}
\]

where \( \mathbf{U} = \int_{z_B}^{z_T} (u, v) \, dz \),

\[
\text{and } z_B = -H(x, y) \quad \text{and} \quad z_T = \eta(x, y, t). \tag{2.3-1b}
\]

where \( H \) is depth of fluid at rest and \( \eta \) is the undulation of the top surface, i.e. the free surface elevation in the case of the ocean, see Fig. 2.3-1.

To evaluate \( w_T \) and \( w_B \), consider the bottom and top surfaces \( z = z_T \) and \( z = z_B \) which are material surfaces (consisting of the same material elements and moving with the fluid flow), i.e.:

\[
\frac{D}{Dt}(z - z_T) = 0, \quad \text{i.e. } w_T = \partial z_T / \partial t + \mathbf{u}^h_T \cdot \nabla h z_T \quad \text{at } z = z_T \tag{2.3-2a}
\]

and \( \frac{D}{Dt}(z - z_B) = 0, \quad \text{i.e. } w_B = \mathbf{u}^h_B \cdot \nabla h z_B \quad \text{at } z = z_B \tag{2.3-2b} \)

Substituting into (2.3-1a):

\[
\partial \eta / \partial t + \nabla_h \cdot \mathbf{U} = 0. \tag{2.3-3}
\]

A depth-averaged velocity \( \mathbf{u} \) can be defined:

\[
\mathbf{u} = \int_{z_B}^{z_T} u_h \, dz / D, \quad D = z_T - z_B = H + \eta. \tag{2.3-4}
\]

Thus, \( \frac{\partial \eta}{\partial t} + \nabla_h \cdot (\mathbf{u} D) = 0; \) or \( D_h(D)/Dt + D \nabla_h \cdot (\mathbf{u}) = 0, \tag{2.3-5} \)

where \( D_h/Dt = \partial / \partial t + \mathbf{u} \cdot \nabla_h \) is the horizontal material derivative based on the 2-dimensional depth-averaged velocity. See Appendix for generalization of (2.3-5) to layer between two isopycnals.

Before integrating the momentum equation (2.2-6b) in a similar way, we need to evaluate \( p \) from the hydrostatic equation \( \partial p / \partial z = -\rho g \) which can be integrated from \( z \) to \( \eta \):

\[
p(x, t)/\rho_o = p_a(x, t)/\rho_o + \int_\eta^z b \, dz; \quad b = g \rho / \rho_o, \tag{2.3-6}
\]

where \( p_a(x, t) = \text{atmospheric pressure} \). Then,
\[ \nabla_h p/\rho_o = \nabla_h p_a/\rho_o + g \nabla_h \eta + \int_z^0 \nabla_h b'. \, dz; \quad b' = gp'/\rho_o. \tag{2.3-7} \]

We now integrate the momentum equation (2.2-6b), \[ \int_z^x \text{[equation 2.2.6b]}, \, dz, \] first noting that:
\[ \mathbf{u} \cdot \nabla \mathbf{u}_h = \nabla \cdot (\mathbf{i}(\mathbf{u}_h) + \mathbf{j}(\mathbf{v}_h) + \mathbf{k}(\mathbf{w}_h)) = \nabla \cdot (\mathbf{u}_h) = \partial (\mathbf{u}_h)/\partial x + \partial (\mathbf{v}_h)/\partial y + \partial (\mathbf{w}_h)/\partial z, \]
in which \( \nabla \mathbf{u} = 0 \) is used, then for each integral term
\[ \int_z^x \left( \frac{\partial \mathbf{u}_h}{\partial t} \right) \, dz = \frac{\partial (\mathbf{u}_h)}{\partial t} - \mathbf{u}_{hT} \cdot \frac{\partial \eta}{\partial t}, \tag{2.3-8} \]
\[ \int_z^x \left( \frac{\partial \mathbf{u}_h}{\partial x} \right) \, dz = \partial (\mathbf{u}D\mathbf{u}) + \int_z^x \mathbf{u}' \mathbf{D} \mathbf{u}' \, dz \, dz - \mathbf{u}_{hT} \mathbf{u}_T \frac{\partial \eta}{\partial x} - \mathbf{u}_{hB} \mathbf{u}_B \frac{\partial \eta}{\partial x} \]
\[ \int_z^x \left( \frac{\partial \mathbf{v}_h}{\partial y} \right) \, dz = \partial (\mathbf{v}D\mathbf{u}) + \int_z^x \mathbf{v}' \mathbf{D} \mathbf{u}' \, dz \, dz - \mathbf{u}_{hT} \mathbf{v}_T \frac{\partial \eta}{\partial y} - \mathbf{u}_{hB} \mathbf{v}_B \frac{\partial \eta}{\partial y} \]
\[ \int_z^x \mathbf{k}_f \times \mathbf{u}_h \, dz = \mathbf{k} \mathbf{D} f \times \mathbf{\bar{u}} \]
\[ \int_z^x \left( \frac{\partial \eta}{\partial z} \right) \, dz = \mathbf{w}_T \mathbf{u}_{hT} - \mathbf{w}_B \mathbf{u}_{hB} = \mathbf{u}_{hT} \left( \frac{\partial \eta}{\partial t} + \mathbf{u}_{hT} \cdot \nabla \eta \right) + \mathbf{u}_{hB} (\mathbf{u}_{hB} \cdot \nabla \mathbf{h}) \]
\[ \int_z^x \mathbf{D} \nabla \eta = \rho \mathbf{p}_a/\rho_o + \mathbf{D} g \nabla \mathbf{h} \eta + \int_z^\eta \mathbf{D} \nabla \mathbf{h} \eta - \mathbf{K}_B \eta \mathbf{u}_h/\partial z \mathbf{d}z \mathbf{d}z \]
\[ \int_z^x \partial (\mathbf{K}_M \mathbf{u}_h/\partial z) \, dz = \mathbf{u}'_h = \mathbf{u}_h - \mathbf{\bar{u}} \]
where \( \mathbf{u}'_h \) is the deviation of (horizontal) velocity from its depth-averaged values, so that \( \int_z^x \mathbf{u}'_h \cdot \mathbf{d}z = 0 \), and \( \mathbf{v}_o \) and \( \mathbf{v}_o \) are kinematic surface (wind) and bottom stresses respectively.

Substitute (2.3-8) into [(2.2-6b),dz]:
\[ \frac{\partial (\mathbf{D} \mathbf{u})}{\partial t} + \partial (\mathbf{u}D\mathbf{\bar{u}})/\partial x + \partial (\mathbf{v}D\mathbf{\bar{u}})/\partial y + \mathbf{k} \mathbf{D} f \times \mathbf{\bar{u}} = \mathbf{D} \nabla \mathbf{p}_a/\rho_o - \mathbf{D} \nabla \mathbf{h} \eta \]
\[ - \int_z^\eta \mathbf{D} \nabla \mathbf{h} \eta - \mathbf{u}'_h \cdot \mathbf{d}z \mathbf{d}z + \mathbf{v}_o - \mathbf{v}_b - \frac{\partial (\mathbf{f}_M \mathbf{u}'_h \, \mathbf{d}z)}{\partial x} - \frac{\partial (\mathbf{f}_M \mathbf{v}'_h \mathbf{D} \mathbf{u}'_h \, \mathbf{d}z)}{\partial y} \tag{2.3-9} \]
Assume \( \mathbf{u}'_h = \mathbf{p}_a = 0 \), and homogeneous fluid (winter well-mixed conditions) \( \rho ' = 0 \), so (exp001):
\[ \frac{\partial \mathbf{\bar{u}}}{\partial t} + \mathbf{\bar{u}} \cdot \nabla \mathbf{\bar{u}} + \mathbf{k} \cdot \mathbf{f} \times \mathbf{\bar{u}} = - \mathbf{D} \nabla \mathbf{h} \eta + (\mathbf{v}_o - \mathbf{v}_b)/\mathbf{D}. \tag{2.3-10a} \]
in which (2.3-5) has also been used. This can be written in vector-invariant form:
\[ \frac{\partial \mathbf{\bar{u}}}{\partial t} + \mathbf{k} (\mathbf{f} + \mathbf{\zeta}) \times \mathbf{\bar{u}} = - \mathbf{D} \nabla \mathbf{h} (\mathbf{g} \eta + \mathbf{f} \mathbf{\bar{u}} / \mathbf{D}) + (\mathbf{v}_o - \mathbf{v}_b)/\mathbf{D}. \tag{2.3-10b} \]
where \( \mathbf{\zeta} = \partial \mathbf{\bar{u}} / \partial x - \partial \mathbf{\bar{u}} / \partial y \). Equations (2.3-5) and (2.3-9) are the depth-averaged equations which are solved in the “external mode” of POM. The code has extra terms which are omitted here, such as the horizontal viscous terms, and terms resulting from the transformation of the \( z \)-coordinate to terrain-following \( \sigma \)-coordinate: \( \sigma = (z-\eta)/(H+\eta) \), see Mellor [2004].
2.4 Simplifications

On shelf seas during winter one can often assume well-mixed conditions so that spatially two-dimensional (xy), depth-averaged equations with $\rho' = 0$ approximately apply. In the followings, we use the simplified equation (2.3-10). Take the curl, $\mathbf{k} \nabla \times (=\nabla_h \times)$ of (2.3-10b) and noting that $\nabla \nabla$ (Any Scaler Function) = 0, we obtain:

$$\mathbf{k} \frac{\partial \zeta}{\partial t} + \nabla_h \times [\mathbf{k} (f+\zeta) \times \mathbf{u}] = \nabla_h \times [(\mathbf{r}^o - \mathbf{r}_b)/D]. \quad (2.4-1)$$

To evaluate the second term, we use the following formula:

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \nabla \cdot \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{A} + \mathbf{B} \times \nabla \mathbf{A} - \mathbf{A} \times \nabla \mathbf{B} \quad (2.4-2)$$

for any 2 vectors $\mathbf{A}$ and $\mathbf{B}$. put $\mathbf{A} = \mathbf{k} (f+\zeta)$ and $\mathbf{B} = \mathbf{u}$:

$$\nabla_h \times [\mathbf{k} (f+\zeta) \times \mathbf{u}] = \mathbf{k} (f+\zeta) \nabla_h \mathbf{u} - \mathbf{u} \nabla_h \mathbf{k} (f+\zeta) + \mathbf{u} \nabla_h \mathbf{k} (f+\zeta) - \mathbf{k} (f+\zeta) \nabla_h \mathbf{u}$$

$$= \mathbf{k} \{(f+\zeta) \nabla_h \mathbf{u} \} - 0 + \mathbf{k} \{ \mathbf{u} \nabla_h (f+\zeta) \} - 0$$

$$= \mathbf{k} \{ -(f+\zeta) \mathbf{D}_h(D)/Dt + \mathbf{u} \nabla_h (f+\zeta) \}, \text{ using (2.3-5) for } \nabla_h \mathbf{u}. \quad (2.4-3)$$

Substitute into $D^t \times (2.4-1)$, and write $\partial \zeta/\partial t \equiv \partial (f+\zeta)/\partial t$, we obtain (homework):

$$D_h(PV)/Dt = \nabla_h \times [(\mathbf{r}^o - \mathbf{r}_b)/D]/D, \quad (2.4-4a)$$

where $PV = (f+\zeta)/D \quad (2.4-4b)$

is called the potential vorticity. Another form of (2.4-4a) which is often useful is obtained by combining (2.4-4a) with the second form of (2.3-5), i.e. $D \times (2.4-4a) + PV \times (2.3-5)$ (homework):

$$D_h(D.PV)/Dt + D.PV \nabla_h (\mathbf{u}) = \nabla_h \times [(\mathbf{r}^o - \mathbf{r}_b)/D]$$

i.e. $\partial (f+\zeta)/\partial t + \nabla_h (\mathbf{u} (f+\zeta)) = \nabla_h \times [(\mathbf{r}^o - \mathbf{r}_b)/D] \quad (2.4-5)$

This form is particularly useful when it is cast into an integral over a closed or semi-enclosed basin (Appendix).

2.5 Conservation of PV

For there is no friction, both at the surface (i.e. wind stress if ocean) and bottom, then the RHS of (2.4-4a) = 0, then:

$$D_h(PV)/Dt = 0, \quad \text{i.e. } PV \text{ constant following a fluid parcel.} \quad (2.4-6)$$
A well-known consequence of PV-conservation is how Rossby wave propagates westward (for $f > 0$), as described in Fig.2.5-1 and Fig.2.5-2 [Gill 1982]. From these figures, we see that the propagation of Rossby wave (or more generally PV-conservation) over a sloping bottom on an “f-plane” ($f = \text{constant}$) is mathematically identical with propagation over a “curving” earth’s surface where $f = 2|\Omega|\sin(\theta)$ varies with latitude $\theta$ but $H = \text{constant}$.

**Conservation of PV:**

- **Shallow**
  
  \[
  \frac{(f+\zeta)}{H} \approx \text{constant}
  \]

  \[\delta H > 0, \delta \zeta > 0\]

  \[\delta H < 0, \delta \zeta < 0\]

  Fig.2.5-1. Topographic Rossby wave, on an “f-plane”: $f = \text{constant} > 0$. For simplicity we assume $|\eta| \ll H$. Fluid parcel’s PV-conservation is then $(f+\zeta)/H = \text{constant}$. Following the blue wave which is perturbed from its initial dashed-line position, parcel “A” is moved to where $D \approx H$ is shallower, i.e. where $\delta H < 0$, therefore $\zeta$ decreases i.e. $\delta \zeta < 0$ in order to keep the parcel’s PV constant.

  Similarly, parcel B acquires a positive relative vorticity $\delta \zeta > 0$. The combined action due to their velocity field (short vectors) is such that the blue wave is moved westward to become the green wave as shown.

- **Deep**
  
  \[\delta f < 0, \delta \zeta > 0\]

  \[\delta f > 0, \delta \zeta < 0\]

  \[f_0\]

  \[f_0 + \Delta f\]

  Fig.2.5-2. Planetary Rossby wave: $f = \text{function of } y$, and is positive; also $H = \text{constant}$. For simplicity we assume $|\eta| \ll H$. Fluid parcel’s PV-conservation is then $(f+\zeta) = \text{constant}$. Following the blue wave which is perturbed from its initial dashed-line position, parcel “A” is moved to where $f$ is larger, i.e. where $\delta f > 0$, therefore $\zeta$ decreases i.e. $\delta \zeta < 0$ in order to keep the parcel’s PV constant. Similarly, parcel B acquires a positive relative vorticity $\delta \zeta > 0$. The combined action due to their velocity field (short vectors) is such that the blue wave is moved westward to become the green wave as shown.
2.6 The equivalence of planetary beta \( \beta \) and sloping bottom topography

To see this simply, expand the LHS of (2.4-4a), and then multiply the resulting equation by \( D \):

\[
D \frac{\partial (PV)}{\partial t} = \frac{\partial \zeta}{\partial t} - \frac{\partial (f + \zeta)}{\partial y} \frac{\partial \eta}{\partial t} - \frac{\partial (f + \zeta)}{\partial D} \frac{\partial H}{\partial y} \frac{\partial}{\partial D} \mathbf{u} \cdot \nabla_{h} D + \mathbf{u} \cdot \nabla_{h} (f + \zeta) = \nabla_{h} \times (\mathbf{v} - \mathbf{v}_{0}) / \partial D \tag{2.6-1}
\]

where \( \mathbf{V} \) denotes the crossed-out terms in \( \mathbf{V} \) are of \( O(\eta / H) \). Note that \( \zeta - f \eta / H \) is the linearized \( PV \). The vector \( \nabla_{h} f \) points poleward, so for convenience (but without loss of generality, since the magnitudes of \( \nabla_{h} f \) and \( f \sqrt{h} \) are usually quite different, as we will see) we can align the isobaths to be zonal with shallower \( H \) towards the pole. Then, (2.6-1) becomes:

\[
\frac{\partial (\zeta - f \eta / H)}{\partial t} + \mathbf{u} \cdot \nabla_{h} (\zeta - f \eta / H) + \mathbf{v} \cdot \nabla_{h} [\mathbf{v} + \mathbf{v}_{0}] \approx \nabla_{h} \times (\mathbf{v} - \mathbf{v}_{0}) / \partial H \tag{2.6-2a}
\]

where \( \beta = \nabla_{h} f = \partial f / \partial y \), and \( \beta_{T} = -(f / H) \partial H / \partial y = -f \partial \ln H / \partial y \). \tag{2.6-2b}

Thus effects of northward topographic shoaling (i.e. \( H \) decreases with \( y \)) in a homogeneous fluid are similar to \( \beta \)-effects in large-scale planetary gyre motions. If \( H \) is exponential, \( \beta_{T} \) is constant and therefore the equivalence is exact:

\[
H = H_{0} \exp(2 \lambda y), \quad \text{where } H_{0} \text{ and } \lambda \text{ are constants}; \tag{2.6-3a}
\]

then, \( \beta_{T} = -f(2 \lambda), \) a constant. \tag{2.6-3b}

Many interpretations in the \( \beta \)-plane can be applied to the case of homogenous fluid motion on a sloping bottom. In particular, if the slope of \( H \) is gentle, i.e. if \( (\lambda y) \) is small:

\[
H \approx H_{0} \{ 1 + 2 \lambda y + O(\lambda y^{2}) \}, \quad | \lambda y | < < 1, \tag{2.6-4a}
\]

and \( \beta_{T} \approx -f(\lambda y) \). \( \lambda y \approx -f(2 \lambda), \) approximately a constant. \tag{2.6-4b}

Thus, \( | \delta H / H | \) is small, and the “\( H \)” on the RHS of (2.6-2) becomes \( H \approx H_{0} \), then (2.6-2) becomes:

\[
D_{h}(\zeta - f \eta / H) / \partial t + \mathbf{v} \cdot [\beta + \beta_{T}] \approx H_{0}^{-1} \nabla_{h} \times (\mathbf{v} - \mathbf{v}_{0}). \tag{2.6-5}
\]

Then the similarity with the \( \beta \)-plane – i.e. wind stress curl, Sverdrup transport etc are apparent.
2.7 Topographic beta $\beta_T$ and the shelf seas

On shelf seas, $\delta H$ can be $\sim 100$ m over the width of the shelf, and $H \approx 100$ m, $|\delta H/H|$ is not small so that the “$H^1$” on the RHS of (2.6-2) cannot be neglected. Then $\nabla_h \times (\tau^o/H)$ is $\neq 0$ even if the wind stress $\tau^o$ may be treated as being spatially uniform because the atmospheric wind varies over a much larger spatial scale, $\sim 1000$ km, than the width of the shelf which is $\sim 100$ km – see Fig.2.7-1.

![Fig.2.7-1](image)

**Fig.2.7-1.** Typical width of the shelf sea and scale of wind stress $\tau^o$. The $\tau^o$ is spatially approximately uniform over the narrow shelf; however, because $H$ varies, $\tau^o/H$ is not uniform and $\nabla_h \times (\tau^o/H) \neq 0$. Note the $z$-scale is greatly exaggerated compared to the horizontal scale, by $\sim 1000:1$.

We compare typical magnitudes of $\beta$ and $\beta_T$:

$$\beta = \partial U/\partial y = \partial(2\Omega \sin(\theta))/\partial y = 2\Omega \cos(\theta) \partial \theta/\partial y = 2 \times 7.292 \times 10^{-5} \text{s}^{-1} \cdot \cos(\theta) / R_{\text{earth}} = 2 \times 10^{-11} \text{m}^1 \text{s}^{-1},$$

(2.7-1a)

where a latitude $\theta = 25^\circ \text{N}$ has been assumed and $R_{\text{earth}} \approx 6371$ km.

$$\beta_T = - (f/H) \partial H/\partial y \approx -(6 \times 10^{-5} \text{s}^{-1} / 50 \text{m}) \cdot (10 \text{m} - 100 \text{m}) / (10^5 \text{m}) \approx 1.2 \times 10^{-9} \text{m}^1 \text{s}^{-1}.$$  

(2.7-1b)

Thus $\beta_T > > \beta$, and the latter can be safely neglected when applying (2.6-2) to the shelf seas:

$$\partial (\zeta - f\eta/H)/\partial t + \vec{u} \cdot \nabla_h (\zeta - f\eta/H) + \vec{v} \beta_T \approx \nabla_h \times [(\tau^o - \tau_b)/H]$$

(2.7-2)

Sufficiently “far” (which needs to be more precisely defined, later) from the coastline, assuming steady state and negligible bottom friction $\tau_b/H$, the main balance is:

$$\vec{v} \beta_T \approx \nabla_h \times (\tau^o/H)$$

(2.7-3)

which for positive $\nabla_h \times (\tau^o/H)$ (as in Fig.2.7-1 or Fig.2.1-1) would suggest a positive $\vec{v}$, i.e. onshore flow towards the coast as sketched in Fig.2.1-1. The $H\vec{v}$ is called the topographic Sverdrup transport (per unit length) in analogy to $\beta$-induced Svedrup transport of the world’s oceans’ gyres.
2.8 Quasi-geostrophic Potential Vorticity Equation

First approximate \( f \approx f_0 \), a constant, in equation (2.7-2):
\[
\frac{\partial(\zeta - f_0 \eta/H)}{\partial t} + \bar{u} \cdot \nabla_h(\zeta - f_0 \eta/H) + \bar{v} \beta_T = \nabla_h \times [(\tau^0 - \tau_b)/H]
\]  
(2.8-1)

The error is small since \( f \) is time-independent and \( \bar{u} \cdot \nabla_h(\eta/H) \) is already small compared to \( \bar{v} \beta_T \) (see III_{h\eta} & III_{\eta} after (2.6-1)). Secondly, for small \( \epsilon = |\zeta/f| \), flows are approximately geostrophic, so that the main balance in momentum equation (2.3-10a) is, with subscript “g” on the velocity denoting “geostrophic”:
\[
f_0 k \times \tilde{u}_g = -g \nabla h \eta.
\]  
(2.8-2)

Take the curl \( \nabla_h \times \) and use \( \nabla_h \times (A \times B) = A \nabla_h B - B \nabla_h A + B \nabla_h A - A \nabla_h B \) for any 2 vectors \( A \) and \( B \):
\[
\nabla_h \tilde{u}_g = 0
\]  
(2.8-3)

which says that strictly \( (f = f_0) \) geostrophic flows are horizontally non-divergent. Take the divergence \( (\nabla_h) \) of (2.8-2) and use \( \nabla_h(A \times B) = B \nabla_h A - A \nabla_h B \):
\[
-f_0 k \cdot \nabla_h \tilde{u}_g = -g \nabla_h^2 \eta \quad \text{or} \quad \zeta = (g f_0) \nabla_h^2 \eta.
\]  
(2.8-4)

A stream function, \( \psi \), can then be defined:
\[
\psi = (g f_0) \eta
\]  
(2.8-5)

so that \( \tilde{u}_g = -\partial \psi/\partial y, \quad \tilde{v}_g = \partial \psi/\partial x, \quad \zeta = \nabla^2 \psi \) and \( \nabla_h \tilde{u}_g = 0. \)  
(2.8-6)

Therefore, in geostrophic flows, the free surface is also a stream function. Since \( f_0 \eta/H = f_0^2 \psi/(gH) = \psi/R^2 \), where \( R = (gH)^{1/2}/f \) is the Rossby radius, equation (2.8-1) becomes,
\[
\frac{\partial (\nabla^2 \psi - \psi/R^2)}{\partial t} + J(\psi, \nabla^2 \psi - \psi/R^2) + \beta_T \partial \psi/\partial x = \nabla_h \times [(\tau^0 - \tau_b)/H]
\]  
(2.8-7)

where \( J(a, b) = \partial a/\partial x \partial b/\partial y - \partial b/\partial x \partial a/\partial y \) for any functions “a” and “b.” This is the quasi-geostrophic (QG) PV-equation for a homogeneous fluid. More generally \( \beta_T \) is “\( \beta + \beta_T \),” and (2.8-7) is:
\[
\frac{(D\psi/Dt)(QGPV)}{D\psi/Dt} = \nabla_h \times [(\tau^0 - \tau_b)/H], \quad \text{QGPV} = f_B + \nabla^2 \psi - \psi/R^2,
\]  
(2.8-8a)

where
\[
D\psi/Dt = \partial \psi/\partial t + \tilde{u}_g \cdot \nabla_h \equiv \partial \psi/\partial t + J(\psi, ..), \quad \text{and} \quad f_B = f_0 + \beta y - f_0 \ln(H).
\]  
(2.8-8b)

Here \( f_B \) is background vorticity, and \( \beta_T = -f. \partial \ln(H)/\partial y \approx -\partial[f_0 \ln(H)]/\partial y, \) from (2.6-2b). Following a fluid parcel advected by the geostrophic flow, the QGPV is conserved if there is no friction. Stretching \( (D\psi/Dt > 0) \) tends to be accompanied by production of cyclonic \( \zeta (D\psi/Dt > 0) \) and/or poleward movement of parcel \( D\psi f_B/Dt = \tilde{v}_g df_B/\partial y > 0, \) and vice versa for squashing. With friction, \( \tau^0 \) and/or \( \tau_b > 0, \) curl of depth-distributed net stress \( (i.e. \nabla_h \times [(\tau^0 - \tau_b)/H]) \) produces source or sink of QGPV, modifying the flow. An example follows.
Suppose (a) steady state; (b) \(\tau^0 \neq 0\) but \(\tau_b = 0\); (c) deep H but gentle slope so that \(|\delta H/H| \ll 1\) and the RHS \(\approx \nabla h \times (\tau^0 - \tau_b) H_0^{-1}\), where \(H_0 = \) constant (some averaged H), also \(\beta_T \approx - (f_o/H_0)(dH/dy)\); (d) \(R \ll L\) so that \(|\nabla^2| \ll 1/R^2\); and (e) \(\beta = 0\). The QGPV equation becomes:

\[-\vec{u}_g \cdot \nabla h (\eta + H) = (\nabla h \times \tau^0)/f_o.\]  

(2.8-9)

Note how the \(\beta_T\) term is combined with \(\eta/R^2\) term to form the LHS. The RHS is recognized as being proportional to the Ekman vertical velocity. Simplify to consider only the \(yz\)-plane (meridional section), and \(\tau^0 = (\tau^{\alpha x},0)\):

\[-\vec{u}_g \partial (\eta + H)/\partial y = - (\partial \tau^{\alpha x}/\partial y)/f_o.\]  

(2.8-10)

Then, for \(H\) that shallows to the north, \(dH/dy < 0\), a positive wind stress curl (RHS > 0) would drive northward depth-averaged geostrophic velocity \(\vec{v}_g > 0\). If on the other hand, \(H = \) constant or \(dH/dy\) is very weak, then northward flow would still result if the sea level drops northward: \(\partial \eta/\partial y < 0\). However, in this case, instead of being fixed by the topography, the ambient vorticity is determined by the flow dynamics itself. The physics is shown in Fig.2.8-1.

**Fig.2.8-1.** Equatorward Ekman transport in the Ekman layer near the surface of the ocean is produced by curl of the eastward wind stress. The stress distribution is such that it produces upwelling and lowest sea level at center (left in the sketch), so that \(\partial \eta/\partial y < 0\). Left of center (not shown), the wind is antisymmetrical blowing westward. The (frictionless) bottom has a slope \(dH/dy\) that is here shown to be also \(< 0\). The resulting \(\partial (\eta + H)/\partial y < 0\) forces a northward geostrophic interior velocity \(\vec{v}_g > 0\) with a volume flux which balances the Ekman transport.

The above is idealized. In reality, other factors also come into play. Bottom friction, for example, would produce a bottom Ekman transport that can cancel the surface Ekman transport, reducing \(\vec{v}_g\) to \(\approx 0\). A rotating vortex with zonal velocity is then produced.
2.9 Free & standing Rossby waves

Consider the case of no forcing: \( \tau^0 = \tau_0 = 0 \). Then equation (2.8-7) becomes homogeneous

\[
\partial (\nabla^2 \psi - \frac{\psi}{R^2})/\partial t + J(\Psi, \nabla^2 \psi - \frac{\psi}{R^2}) + \beta \tau \partial \psi/\partial x = 0
\] (2.9-1)

which admits a free-wave solution which we now describe. Consider fluctuations (with stream function \( \phi \)) embedded in a steady \( x \)-directed (zonal) velocity \( U \) (Fig.2.9-1):

\[
\psi = -U y + \phi, \quad U = \text{constant.}
\] (2.9-2)

Then,
\[
(\partial/\partial t + U \partial/\partial x)(\nabla^2 \phi - \frac{\phi}{R^2}) + (\beta + U/R^2) \partial \phi/\partial x + J(\phi, \nabla^2 \phi - \frac{\phi}{R^2}) = 0
\] (2.9-3)

We look for a wave solution, substitute the following to equation (2.9-1):

\[
\phi = A \exp[i(kx + ly - \omega t)].
\] (2.9-4)

\[
(-\omega + Uik \{-k^2 + 1/R^2\} + (\beta + U/R^2)ik = 0 \quad \text{(Note: "J" = 0)}
\]

or

\[
\omega[K^2 + 1/R^2] = -(\beta + U/R^2)k + Uk[K^2 + 1/R^2], \quad \text{where } K^2 = k^2 + l^2,
\]

or

\[
\omega = -\beta k (K^2 + 1/R^2) + Uk [1 - R^2/(K^2 + 1/R^2)]
\]

or

\[
\omega = k(UK^2 - \beta)/(K^2 + 1/R^2).
\] (2.9-5)

**Fig.2.9-1.** Rossby wave in a steady streaming zonal flow with velocity \( U \) shown as being positive.

The phase speed in the \( x \)-direction is:

\[
C_x = \omega/k = (UK^2 - \beta)/(K^2 + 1/R^2) = U - (\beta + U/R^2)/(K^2 + 1/R^2)
\] (2.9-6)
The wave becomes *stationary*, i.e. \( C_x = 0 \), when \( \beta K^2 - \beta = 0 \), or,

\[
K_s = (\beta/U)^{1/2}
\]

In the atmosphere, \( U = U_{\text{atmos}} \approx 40 \text{ m s}^{-1} \) in the jet-stream at \( \sim 400 \) hPa. Then, the stationary wavelength:

\[
\lambda_{\text{atmos}} = 2\pi/K_s = 2\pi(U_{\text{atmos}}/\beta)^{1/2} \approx 6.0(40/(2\times10^{-11}))^{1/2} \approx 8000 \text{ km},
\]

see Fig.2.9-2.

---

**Fig.2.9-2.** Climatological pressure (anomaly) map for the northern hemisphere from Lau [1979] showing standing Rossby wave pattern, with low pressures downwind of the Himalays & Rockies.

For topographic Rossby wave in exp001, from (2.7-1b), \( \beta_T = -(f_o/H)\partial H/\partial y \approx (f_o/H)(40\text{m})/175\text{km} \) (Fig.2.1-1), where 40 m is the difference in water depth across the gray-shaded study region whose width \( \approx 175 \text{ km} \). Take \( f_o \approx 6\times10^{-5} \text{ s}^{-1} \) and \( H \approx 50 \text{ m} \), \( \beta_T \approx 2.7\times10^{-10} \text{ m}^{-1} \text{ s}^{-1} \). Then,

\[
\lambda_{\text{ocean}} = 2\pi/K_s = 2\pi(U_{\text{ocean}}/\beta_T)^{1/2} \approx 6.0(0.2/(2.7\times10^{-10}))^{1/2} \approx 170 \text{ km},
\]

which is approximately the wavelength seen in the numerical solution shown in Fig.2.9-3.
Fig. 2.9-3. Exp001 modeled sea-surface height (color & dark contours, dotted if negative, m) and currents (vectors, red for u < 0 in m s⁻¹) for flow over a ridge inside the gray-shaded region of Fig 2.1-1. White dashed contours are isobaths = 70, 50, 40 & 30 m (on the ridge, radius ~ 30 km). The “x” and “y” axes are shown as °E and °N, though their values are arbitrarily chosen.

2.10 Comparison between model and theory


Homework:

2.1: Derive (2.4-4) and (2.4-5).

2.2: Set U = 0 in (2.9-6), then plot ω as a function of k, and discuss its characteristics.

2.3: Let l = 0 but U ≠ 0 in (2.9-6), then discuss the dispersion curve ω-k plot for (a) U > 0 and (b) U < 0.

2.4: Modify mpiPOM exp001 to produce U < 0 and write a report for your model calculations for both HW(2.3a) and HW(2.3b) and discuss the results.

2.11 Barotropic Instability

We write (2.9-1) as (the topographic βₜ set = β):

$$\partial (\nabla^2 \psi + \beta \psi - \psi/R^2)/\partial t + J(\psi, \nabla^2 \psi + \beta \psi - \psi/R^2) = 0$$

(2.11-1)
We will now find the condition of the type of horizontal shear that can produce barotropic instabilities. Let the background (or initial) zonal flow be $U_o(y)$, with stream function $\Psi(y)$:

$$U_o(y) = -\partial\Psi/\partial y \quad \text{(note that } V_o = \partial\Psi/\partial x = 0)$$  \hspace{1cm} (2.11-2)

Introduce a small perturbation “$\phi$”, so that the total, time-dependent stream function $\psi$ is:

$$\psi(x,y,t) = \Psi(y) + \phi(x,y,t)$$  \hspace{1cm} (2.11-3)

The function $\phi$ is perturbation to the background state $\Psi$; it represents the structure of the evolving perturbation field. Substitute (2-11.3) into (2-11.1):

$$(\partial/\partial t + U_o\partial/\partial x - \phi_y\partial/\partial x + \phi_x\partial/\partial y)\{q + \partial^2\Psi/\partial y^2 + \beta y - \Psi/R^2\} = 0,$$

or

$$(\partial/\partial t + U_o\partial/\partial x)q + J(\phi,q) + \phi_x\partial\Pi_o/\partial y = 0$$  \hspace{1cm} (2.11-4)

where $q(x,y,t)$ is the perturbation PV:

$$q = \nabla^2\phi - \phi/R^2$$ \hspace{1cm} (2.11-5a)

and

$$\Pi_o = \partial^2\Psi/\partial y^2 + \beta y - \Psi/R^2$$ \hspace{1cm} (2.11-5b)

is the background PV, and

$$\partial\Pi_o/\partial y = -\partial^2 U_o/\partial y^2 + \beta + U_o/R^2$$ \hspace{1cm} (2.11-5c)

How does the structure of $U_o(y)$ determine the evolution of the perturbation field $\phi$? That is, given a particular background state $U_o(y)$, will the perturbation $\phi$ “injected” on the flow grows or decays? If it grows, then the background state is unstable with respect to the perturbation $\phi$. To show that $U_o$ is stable, we must check all possible $\phi$'s. On the other hand, to show instability, we only need to find one perturbation to which the background state $U_o$ is unstable.

**Linear Stability Analysis:**

We assume that $|\phi| << 1$ so that the $J(\phi,q)$ in (2-11.4) is neglected:

$$(\partial/\partial t + U_o\partial/\partial x)q + \phi_x\partial\Pi_o/\partial y = 0$$  \hspace{1cm} (2.11-6)

A perturbation energy equation can be derived (see below) and it can be shown that:

$$\int\int p_s (\partial\Pi_o/\partial y)(\partial<\eta^2>/\partial t) \, dydz = 0$$  \hspace{1cm} (2.11-7)

where $\eta$ is the meridional displacement of fluid elements defined by:

$$\partial\eta/\partial t + U_o\partial\eta/\partial x = \partial\psi/\partial x$$ \hspace{1cm} (2.11-8)

and $<.>$ means zonal-averaging. From (2-7.7), we see that if there is to be a growth in the displacement of fluid elements in time, i.e. if $\partial<\eta^2>/\partial t > 0$, then $\partial\Pi_o/\partial y$ must be somewhere positive and somewhere else negative in the $yz$-plane, or $\partial\Pi_o/\partial y$ can also be identically zero everywhere. In other words, it is necessary that $\partial\Pi_o/\partial y$ vanishes on a line in the $yz$-plane. Clearly, this condition (i.e. that $\partial\Pi_o/\partial y$ must vanish somewhere) is not sufficient.
The Perturbation Energy Equation Derived From the Linear Equation (2.11.6):

Multiply (2-11.6) by ϕ:

\[
\phi \partial q / \partial t + \phi U_0 \partial q / \partial x + \phi \phi_0 \partial \Pi_y / \partial y = 0
\]  

(2.11-9)

Although our variables are not function of “z”, the following general derivations are valid if “q”, U_0, and \( \partial \Pi_y / \partial y \) also depend on “z”.

Integrate (2.11-9) over the fixed volume

\[
\int_V dV = \iiint dxdydz
\]  

(2.11-10)

where the integral is from \( y = y_{South} \) to \( y = y_{North} \), \( z = z_{Bot} \) to \( z = z_{Surf} \) and x-periodic, and repeatedly make use of the integration-by-part formula

\[
\int u dv = [uv] - \int v du, \text{ for any function } u \text{ and } v.
\]  

(2.11-11)

The integral of the 3rd term of (2.11.9) is zero:

\[
\int_V (\phi_0 \phi \Pi_y / \partial y).dV = \iiint (\partial \Pi_y / \partial y)d(\phi^2 / 2).dydz = \iiint [\phi^2 / 2]. \partial \Pi_y / \partial y.dydz = 0
\]  

where \( [Q]_1 = [Q]_2 - Q|x| = 0 \) for any periodic function “Q.”

The integral of the 1st term of (2.11.9) is, term by term:

(1a):

\[
\iiint \phi \partial q / \partial t.dV = \iiint [\phi \phi_0 x].dydz = \iiint [\phi_0 x].dydz - \iiint [\phi_0 x].dxdydz = 0 - \iiint (\phi_0^2 x).dV
\]

\[
\partial / \partial t \{ \iiint <\phi_0^2 x> dxdz \} .L_x
\]

(2.11.12a)

where \( <Q> = \int Q dx / L_x \), i.e. zonal averaged with \( L_x \) = zonal length of the x-periodic channel.

(1b):

\[
\iiint \phi \partial q / \partial t.dV = \iiint [\phi \phi_0 y]. dx = \iiint [\phi_0 y].\phi_0 x]. dx = \iiint [\phi].\phi_0 x]. dx, \text{ since } \partial \phi / \partial x = 0 \text{ at } y_{South} \text{ and } y_{North}. \text{ Next, from the x-momentum equation, we have after } <..> = \int .. dx / L_x:
\]

\[
\partial <u_o>/ \partial t = f_o<v_1> + \beta_o y<v_o> - \partial <u_o v_o>/ \partial y = f_o<v_1> - \partial <u_o v_o>/ \partial y, \text{ since } <v_o> = 0
\]

because \( v_o \) is geostrophic. At both \( y = y_{South} \) and \( y_{South} \), \( v_1 \) and \( v_o \) are zero, so that:
\[ \partial \mathbf{u}_y / \partial t = - \phi_y, \] at \( y = y_{\text{North}} \) and \( y_{\text{South}} \); i.e. \( \int \phi_y \partial \mathbf{u}_y \partial x \, dx = 0 \), and therefore:

\[ \int [ \phi \phi_y ] \partial x \, dx = 0, \text{ proved.} \]

(1c):

\[ \int \int \left[ \phi \partial (S^{-1} \phi_z) / \partial t \right] dV = \int \int \left[ \phi \partial (S^{-1} \phi_z) / \partial t \right] dV = 0 - \int \int S^{-1} \phi_z, \text{proven.} \]

In above, \( \int [ \phi \phi_y ] \partial x \, dx = 0 \) because \( \phi_z \sim w_1 = 0 \) at \( z = z_{\text{Bot}} \) and \( z = z_{\text{Surf}} \) (no friction, otherwise there is Ekman pumping).

So, the integral of the 1st term of (2.11.11) is:

\[ \int \int \phi \partial \mathbf{q} / \partial t \partial x \, dV = - \partial / \partial t \{ \int [ \phi \phi_x^2 ] + \int [ \phi \phi_y^2 ] + S^{-1} \int [ \phi \phi_z^2 ] \} / 2 \, dydz \cdot L_x \]  (2.11.12)

The integral of the 2nd term of (2.11.9) is:

\[ \int \int \phi \partial \mathbf{q} / \partial x \, dV = \int \int \left\{ \phi \partial (U_0 \phi_x) / \partial x \right\} dV = \int \int \left\{ \phi \partial (U_0 \phi_x) / \partial x \right\} dV 
\]

after applying periodicity \( [ \phi_x^2 / 2 ]_x = 0 \) and boundary conditions \( \phi_x = 0 \) in \( \left[ \phi_y U_0 \phi_x \right]_y = 0 \), and \( \phi_z = 0 \) in \( \left[ S^{-1} \phi_z U_0 \phi_x \right]_z = 0 \) (no friction, so no Ekman pumping at \( z = 0 \) & \( z = -1 \)).

Then, expanding \( (U_0 \phi_x) \) and \( (U_0 \phi_z) \) in the above integrals, we obtain:

\[ \int \int \left\{ \phi \partial (U_0 \phi_x) / \partial x \right\} dV = \int \int \left\{ \phi \partial (U_0 \phi_x) / \partial x \right\} dV = \int \int \left\{ \phi \partial (U_0 \phi_x) / \partial x \right\} dV 
\]

The 2nd and 4th terms are zero because of periodicity.

So, the integral of the 2nd term of (2.11.11) is:

\[ \int \int \left\{ \phi \partial (U_0 \phi_x) / \partial x \right\} dV = \int \int \left\{ \phi \partial (U_0 \phi_x) / \partial x \right\} dV = \int \int \left\{ \phi \partial (U_0 \phi_x) / \partial x \right\} dV 
\]

Finally, combining (2.11.12) and (2.11.13) then gives \( \int \int (2.11.11) dV \rightarrow \)

\[ \partial / \partial t \{ \int [ \phi \phi_x^2 ] + \int [ \phi \phi_y^2 ] + S^{-1} \int [ \phi \phi_z^2 ] \} / 2 \, dydz = 0 \]

(2.11.14)
Chapter 3: Geostrophic adjustment, front & baroclinic instability

This chapter deals with experiment mpiPOM exp002: ([ftp://profs.princeton.edu/leo/mpipom/atop/tests/exp002/]). In contrast to “exp001” in which all variables are depth-averaged with constant-density fluid and the experiment is two-dimensional in the xy-plane, the flow is now three-dimensional (xyz) and there are density stratifications both in vertical as well as horizontal directions. As in the previous chapter, we first describe the set-up, physics and goal of the experiment. Next we describe the geostrophic adjustment process. We then consider the Eady model of baroclinic instability. We then analyze the results to check against the theory, discuss discrepancies between theory and experiment, and finally, analyze the results to understand the discrepancies.

Fig.3.1-1. mpiPOM exp002: x-periodic channel on an f-plane (f = 6×10^{-5} s^{-1} at 24.3°N) with initially warmer water in the southern half and cooler water in the northern half of the channel. Right insert illustrates the idea of x-periodicity with an annulus channel, which approximates the straight channel as the ratio of radius to width of the annulus channel, $R_{ann}/r$, becomes large. Scale is approximate. Water depth H is constant = 50 m. Grid sizes are $\Delta x = \Delta y = 1.112$ km and $\Delta z = 1$ m.

3.1 Set-up, physics and goal

A channel of x-length = 210 km and y-length = 50 km has walls along its southern and northern sides and is x-periodic, meaning that flow disturbance exiting the eastern or right-hand end reenters the channel at the western end, and/or vice versa (Fig.3.1-1). At time $t = 0$, water in the southern half of the channel is warmer, temperature $T = 25$ °C and is cooler with $T = 15$ °C in the northern half; the salinity is uniform, $S = 35$ psu. The surface is flat (sea-surface height SSH = $\eta = 0$) and velocity $u = 0$. The initially straight front separating the two fluids of different densities is then allowed to adjust. Figure 3.1-2 shows $\eta$ and surface currents at days 1, 3, 7 and 10. At $t = 0$, $\eta = 0$, but model adjusts rapidly such that by day1 $\eta$ is higher in the south, $\approx 0.03$ m, and lower in the north, $\approx -0.03$ m. The surface flow is eastward ($u > 0$) but a northward current ($v > 0$) near the mid-channel remains. At day3, the flow is eastward. At day7 small-amplitude wave develops. The system is unstable, and by day10, the wave has grown to produce a meandering flow. The goal is to study the adjustment as well as the instability processes.
Fig. 3.1-2. mpiPOM exp002: model SSH (color, cm; white contour is SSH = 0) and surface currents (scale is shown at lower left), from top to bottom: at days 1, 3, 7 and 10.
3.2 Why is $\eta$ higher in the south than north?

Integrating the hydrostatic equation (2.2-4) $\partial pl/\partial z = -pg$ from $-H$ to $\eta$ (see Fig.2.3-1):

$$p_a = p_b - \int_{-H}^{\eta} \rho g dz' = p_b - \rho g(\eta + H),$$  \hspace{1cm} (3.2-1)

where $p_a$ is the atmospheric pressure at $z = \eta$, $p_b$ is pressure at bottom $z = -H$, and in the second equality density $\rho$ is assumed constant. At $t = 0$, $\rho = \rho_S$ is constant in the south and similarly $\rho = \rho_N$ ($> \rho_S$) is a constant in the north, where subscripts “N” and “S” denote north and south respectively. The hydrostatic adjustment (3.2-1) is established nearly instantaneously as soon as the simulation starts. Write (3.2-1) for north and south, subtracting and rearranging:

$$\eta_N' = (\rho_S/\rho_N)(1 + \eta_S') - 1 + \Delta(p_b' - p_a')$$  \hspace{1cm} (3.2-2a)

where $\eta' = \eta/H, \quad p' = p/(gH\rho_N)$ and $\Delta = (..)_N - (..)_S$ (i.e. North minus South). (3.2-2b)

Since $(\rho_S/\rho_N) \approx 1$, we write $(\rho_S/\rho_N)\eta_S' \approx \eta_S'$; also $\Delta p_a' = 0$. (In any case $p_b' >> p_a'$). Therefore:

$$\Delta \eta' \approx -\Delta \rho/\rho_N + \Delta p_b',$$  \hspace{1cm} (3.2-3)

which shows that, since $\Delta \rho > 0, \Delta \eta' < 0$, provided that contribution from $\Delta p_b'$ is small; i.e. free surface is higher in the south than north.

The density $\rho$ is a function of $T$, $S$ and $p$, and is computed in mpiPOM using the equation of state which is complicated (subroutine dens), but for shallow $H (<500 \text{ m})$ a rough estimate is:

$$\Delta \rho/\rho \approx -\alpha \Delta T; \quad \text{where} \quad \alpha = -\rho^{-1}(\partial \rho/\partial T)_{S,p} (S,p \text{ mean “at constant } S \text{ & } p”)$$  \hspace{1cm} (3.2-4)

$$\approx -2 \times 10^{-4} \Delta T \quad \text{for } T \approx 10-25 \text{ }^\circ \text{C} \text{ (see Gill 1982, Table A3.1).}$$

In the model, $\Delta T \approx -10 \text{ }^\circ \text{C}$, so that $\Delta \rho/\rho_N \approx +2 \times 10^{-3}$. Then,

$$\Delta \eta' \approx -2 \times 10^{-3} + \Delta p_b',$$  \hspace{1cm} (3.2-5)

which is close to the model $\Delta \eta'_{\text{model}} \approx -0.07 m/H \approx -1.4 \times 10^{-3}$ at day1 (Fig.3.1-2). Thus $\Delta p_b' \approx +6 \times 10^{-4}, \text{ i.e. bottom pressure is larger in the north than south, and it drives a southward bottom flow ($v < 0$) which within a few hours may be approximated by:}$

$$rv \approx -\rho^{-1}\partial p/\partial y$$  \hspace{1cm} (3.2-6)

where $r \approx 2 \times 10^{-3}/H = 4 \times 10^{-5} \text{ s}^{-1}$ is the model’s linearized bottom friction coefficient, or it can just be the inverse time scale ~1/6hours. Therefore, taking $\Delta y \approx 10 \text{ km}$, the width of the front,

$$v \approx -r^{-1}[gHDp_b/\Delta y] \approx -r^{-1}[500 \times 6 \times 10^{-4}/(10^4)] = -r^{-1}[3 \times 10^{-5}] \approx -0.75 \text{ m s}^{-1},$$  \hspace{1cm} (3.2-7)

which is close to the mid-channel $v$ near the bottom in the first few hours of simulation (Fig.3.2-1).

Fig.3.2-1. bottom speed (color) and currents at t = 6 hours. (homework_exp002_wj.doc).
In summary, higher $\eta$ in the south than north is produced by warm and cold fluids and is a direct consequence of hydrostatic. Bottom pressure difference is small but not zero, and it forces southward bottom flow in the early hours of the model simulation. At later times, Coriolis forces the bottom flow to also veer westward, and the system undergoes inertial oscillations.

**Homework 3.2.1:** Analyze (plot) model results to verify the analyses of section 3.2.

### 3.3 Flow without rotation

To understand how the system evolves into one that has current flowing in the direction parallel to the interface of warm and cold fluids in 1~3 days – i.e. $x$-directed currents (Fig.3.1-2), we first study adjustment of the system without rotation. Without rotation, the warmer water of higher free surface in the south would surge northward over the cold water in the north which in turn moves southward beneath the warm fluid. The fluid would slosh back and forth. Given sufficient time, all motions cease and, assuming an ideal case of no heat exchange with the surrounding, the system settles to a rest state with warm water overlying cooler water.

**Homework 3.3.1:** Conduct a model experiment which describes the above fluid adjustment without rotation. Integrate the model to a steady state, and describe how the free surface undergoes adjustment and the system comes to the rest state.

**Homework 3.3.2:** Repeat the above experiment but this time simplify the problem to one-dimensional as follows. Turn off the warm-cold fluid configuration and make the fluid homogeneous. Then make $\Delta x$ very large, say $10^6$ m so effectively the problem becomes independent of $x$. Then initialize the elevation ($\eta$) to be higher on the southern half and lower on the northern half. Again integrate the model to a steady state, and describe how the free surface undergoes adjustment and the system comes to the rest state.

We can analyze the flow described in Homework 3.3.2 with a simple model; the same model can then be extended to the case with rotation. The continuity and momentum equations are from (2.3-5) and (2.3-10b):

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (H u) = 0$$

$$\frac{\partial u}{\partial t} + f k \times u = -g \nabla \eta$$

where for simplicity the “overbar” denoting depth-averaging and subscript “h” on $\nabla_h$ for horizontal grad-operator are omitted. Also, the equations have been linearized so that $\nabla (Du) \approx \nabla (Hu)$ in the continuity equation, and terms $\zeta \times \bar{u}$ and $\frac{\left|\bar{u}\right|^2}{2}$ (or $\bar{u} \cdot \nabla \bar{u}$) are dropped in the momentum equation.

In one spatial dimension ($y$), the continuity equation is
\[ \frac{\partial \eta}{\partial t} + \frac{\partial (H \nu)}{\partial y} = 0. \]  

(3.3-1a)

An easy way to also derive the momentum equation when \( f = 0 \) is to apply the Newton’s Law for the elemental control volume \( \Delta x \Delta y \Delta z \), see Fig.3.3-1, which then gives, in one spatial dimension:

Mass \times \text{Accel.} = \text{Pressure Force} 
\[ \rho_c \Delta x \Delta y \Delta z \times \frac{\delta v}{\delta t} = (p_A - p_B) \Delta x \Delta z \] 
\[ \frac{\delta v}{\delta t} = -\delta p / \delta y; \quad \text{i.e.} \quad \frac{\delta v}{\delta t} = -g \frac{\partial \eta}{\partial y} \]  

(3.3-1b)

Fig.3.3-1 An illustration of the hydrostatic relation for (A) 1-layer fluid and (B) 2-layer.

For \( H = \) constant, we can derive the following simple wave equation \( (c^2 = gH) \):
\[ \frac{\partial^2 \eta}{\partial t^2} - c^2 \frac{\partial^2 \eta}{\partial y^2} = 0 \]  

(3.3-2)

The “Step” Propagation problem:

Suppose that at \( t=0, \nu=0, \) and \( \eta = G(y) \)  

(3.3-3)

Then the solution to (3.3-2) for \( H=\)constant is:
\[ \eta = [G(y+ct) + G(y-ct)]/2 \]  
\[ \nu = -(g/c) [G(y+ct) - G(y-ct)]/2 \]  

(3.3-4) (3.3-5)

where (3.3-5) follows by using (3.3-4) in (3.3-1b).

Step propagation: Suppose \( G(y) = -\eta_o \text{ sgn}(y) \)  

(3.3-6)

then, \( \eta = -\eta_o [\text{sgn}(y+ct)+\text{sgn}(y-ct)]/2 \)  
and \( \nu = (\eta_o g/c)[\text{sgn}(y+ct)-\text{sgn}(y-ct)]/2 \)  

(3.3-7a) (3.3-7b)

See figure 3.3-2.

The kinetic and potential energies per unit horizontal area are:
\[ \text{KE/area} = \int \rho \nu^2/2 \, dz \approx \rho H (\eta_o g/c)^2/2 = \rho g \eta_o^2/2 \]  

(3.3-8a)
PE/area = \int \rho g z \, dz = \rho g [\eta_0^2 - H^2]/2 
(3.3-8b)

(\text{where } \int \text{ is from } z=-H \text{ to } z=\eta).\n
The perturbation PE/area is then \rho g \eta_0^2/2 = KE/area. Therefore, the initial PE/area due to \eta = -\eta_0 \ sgn(y) at t=0 is all converted to KE/area to generate current in the direction of “high pressure \eta to low \eta,” i.e. direction of \partial \eta/\partial y.

\textbf{Fig.3.3-2} Step propagation. At “A”, contribution from the y+ct characteristic is -\eta_0/2 and from the y-ct characteristic is also -\eta_0/2, total solution for \eta from (3.3-7a) is the sum = -\eta_0; for v the solution from (3.3-7b) is their difference = 0. At “B”, contribution from the y+ct characteristic is -\eta_0/2 but from the y-ct characteristic is +\eta_0/2, total solution for \eta is their sum = 0; for v the solution is their difference = \eta_0g/c > 0.

Homework 3.3.3: Modify the model code for Homework 3.3.2 to design an experiment for the above “step propagation.” Note that now there are no solid boundaries and the step-waves must be allowed to propagate to infinity on left and right. Devise methods to do this. Try (i) open boundary conditions; (ii) sponges to absorb; (iii) gradually increasing \Delta y to left and right; or a combination of these methods.

\textbf{Block propagation (fig.3.3-3):}

At t=0, \text{G}(y) = \eta_0 \text{ for } |y| < L, \text{ and } =0 \text{ for } |y|>L. 
(3.3-9)

In this case, the solution is most easily obtained graphically keeping in mind equations 3.3-4 and 3.3-5, as shown in fig.3.3-3.
At “A- & A+”, there are equal contributions from both the $y^+ct$ and $y^-ct$ characteristics, so that $\eta = \eta_o$ and $v=0$ from equations (3.3-4) and (3.3-5). At “B+” the $y^-ct$ characteristic contributes $\eta_o$ but the $y^+ct$ characteristic contributes “0” so that $\eta = \eta_o/2$ and $v=-(g/c)(0-\eta_o)/2=+ \eta_o g/2c$. At “B-” the $\eta$ is also $\eta_o/2$ but the $v$ reverses sign $= - \eta_o g/2c$. At “C- & C+” both $\eta$ and $v$ are zeros.

The $\text{PE}/y$-length at $t = 0$ is:

$$\text{PE}(0) = \int \int \rho g z dydz = 2L \rho g [(z^2/2)|_{z=\eta} - (z^2/2)|_{-H}] \quad (3.3-10)$$

where $\int \int$ is from $-H$ to $z=\eta$, and from $-L$ to $+L$. Therefore,

$$\text{Perturbation PE} = \rho g \eta_o^2 L \quad (3.3-11)$$

At time $t = L/c$ (see Fig.3.3-3), two blocks of length 2L but half height $\eta_o/2$ begin to move away from each other, to the left and right. For $t > L/c$, each block of height $\eta_o/2$ has (perturbation) $\text{PE} = \rho g (\eta_o^2 /4)/2 \times 2L = \text{PE}(0)/4$. The KE of each block is $H2L \rho v^2/2 = HL \rho \eta_o^2 g^2/4c^2 = \rho g \eta_o^2 L/4 = \text{PE}(0)/4$. Therefore, total (KE+PE) energy is $\text{PE}(0)/2$ per block, i.e. $\text{PE}(0)$ for 2 blocks. In this case, for each block, $1/2$ of the total energy is in PE and the other $1/2$ is in KE.

**Homework 3.3.4**: Modify the model code for Homework 3.3.3 to simulate the step-propagation of Fig.3.3-3.

For both Homeworks 3.3.3 and 3.3.4, it is necessary that model simulations retain the sharp jumps at the edges of the propagating blocks. In these cases, the cell Reynolds number $R_c$ (see equation H.1.6-3b) is likely to exceed 2 (it depends on the diffusivity, $\alpha$, that you pick), so oscillations will likely exist. Find out how to eliminate or reduce the oscillations.
3.4 Flow with rotation: geostrophic adjustment

In early times <5 days, the surface flow adjusts from south-north to west-east (Fig. 3.1-2). The flow turning is due to rotation and the corresponding adjustment is called geostrophic adjustment which we now study by adding rotation to the “step” model of the previous section (Fig. 3.3-2). The linearized momentum and continuity equations are:

\[
\begin{align*}
\frac{\partial u}{\partial t} - fv &= -g\frac{\partial \eta}{\partial x}, \\
\frac{\partial v}{\partial t} + fu &= -g\frac{\partial \eta}{\partial y}, \\
\frac{\partial \eta}{\partial t} + H(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) &= 0.
\end{align*}
\]

(3.4-1a, b, and c)

Then,

\[
\frac{\partial^2 \eta}{\partial t^2} + Hf\zeta - c^2\nabla^2 \eta = 0 \quad (c^2 = gH & \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}).
\]

(3.4-3)

\[
\nabla x(3.4-1): \quad \frac{\partial (\zeta - f\eta/H)}{\partial t} = 0
\]

(3.4-4)

The quantity \( Q' = \zeta/H - f\eta/H^2 \) is called the perturbation potential vorticity (PV).

(3.4-5)

\[
(3.4-4 & 5) \Rightarrow Q'(x,y,t) = Q'(x,y,0)
\]

(3.4-8)

We consider the “step” problem again. The initial step is:

\[
\eta = -\eta_o \text{sgn}(y) \text{ and } (u,v) = (0,0) \text{ at } t = 0.
\]

(3.4-9)

(3.4-4) gives: \( \zeta = \frac{\eta}{H} = HQ'/f = \text{initial value} = HQ'(x,y,0)/f = (\eta_o/H)\text{sgn}(y) \)

(3.4-10)

Then (3.4-3): \( \frac{\partial^2 \eta}{\partial t^2} - c^2\nabla^2 \eta + f^2\eta = -fH^2Q'(x,y,0) = -f^2\eta_o\text{sgn}(y) \)

(3.4-11)

If a steady solution exist, then (3.4-1a, b) leads to geostrophic degeneracy because:

\[
u = -(g/f)\frac{\partial \eta}{\partial y} \text{ and } v = (g/f)\frac{\partial \eta}{\partial x},
\]

(3.4-12)

and therefore the continuity equation (3.4-2) is identically satisfied:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

(3.4-13)

i.e. it cannot be used as the 3rd equation for the 3 unknowns \((u, v, \eta)\). In fact, a stream function \( \psi \) can be defined so that it satisfies (3.4-13):

\[
u = -\frac{\partial \psi}{\partial y} \text{ and } v = \frac{\partial \psi}{\partial x}, \text{ where } \psi = (g/f)\eta \text{ (unit = m}^2 \text{s}^{-1}).
\]

(3.4-14)
The required 3rd equation is in fact the conservation of PV equation (3.4-8). This says that PV is in a “steady state” but that the velocity field (expressed by $\zeta$) and height field ($\eta$) are closely connected to their initial distribution. In other words, rotational flow has “memory” which in turn governs the subsequent evolution of the flow field.

For geostrophic flow, the vorticity $\zeta$ is:

$$\zeta = (g/f) \nabla^2 \eta$$

(3.4-15)

The equation of conservation of potential vorticity (PV) (3.4-8) is then:

$$Q' = \zeta/H - f\eta/H^2 = [(g/f) \nabla^2 \eta]/H - (f/H^2)\eta = Q'(x,y,0)$$

therefore,

$$-c^2 \nabla^2 \eta + f^2 \eta = -fH^2 Q'(x,y,0)$$

(3.4-16)

which is just the steady-state form of (3.4-11). Setting $\partial/\partial x = 0$ for our problem:

$$-c^2 d^2 \eta/dy^2 + f^2 \eta = -f^2 \eta_{io} \text{sgn}(y)$$

(3.4-17)

So that, for solution that is continuous at $y = 0$, where $\eta = 0$, we have:

$$\eta/\eta_{io} = e^{-y/R} - 1 \quad \text{for } y > 0$$

(3.4-18a)

$$= -e^{-y/R} + 1 \quad \text{for } y < 0$$

(3.4-18b)

where $R = c/|f|$ is called the Rossby radius.

(3.4-19)

Velocity $(u,v)$ is from (3.4-12)

$$u = [g\eta_{io} / (fR)] e^{-y/R} \quad \text{and} \quad v = 0.$$  

(3.4-20)

The adjustment process is different with rotation, i.e. if the channel is placed on a rotating table. The idea can be described by a laboratory experiment (Fig.3.4-1). The angular momentum is:

$$r V_\theta = \text{constant} = r (v_0 + \Omega r) = \Omega r_1^2$$

(3.4-21)

Here, $V_\theta =$tangential velocity in the inertial frame of reference and $v_0$ is the tangential velocity in the frame of the rotating disc (tank), and $r_1 =$ radius of disc where $v_0 = 0$. Solving (3.4-21) for $v_0$:

$$v_0 = \Omega (r_1^2 - r^2)/r.$$  

(3.4-22)

Thus $v_0$ becomes very large near the center of the disc – as seen in the animation below.
Fig. 3.4-1. Laboratory experiment: effects of rotation on the trajectory of a particle (float) placed initially near the rim of the rotating circular dishpan for slow (left; GFD3_slow_rotation.mpeg) and fast (right; GFD3_fast_rotation.mpeg) rotations. The water is (very slowly) drained in the center under the dishpan so there is a weak radial velocity into the center, and this purely radial flow is the only one that exits for zero rotation $\Omega = 0$, and one expects the float to also move inward along the radial direction. For $\Omega \neq 0$ (positive counterclockwise as shown), the float spirals counterclockwise into the center. For fast rotation (right), the float circles around the center before falling into it. With slow rotation (left), the trajectory of the float is more radial and it falls rather more quickly into the center. In the limit of very high rate of rotation, i.e. very, very large $\Omega$, the float would execute many rotations before falling into the center. From Marshall and Plumb [2010].

Fig. 3.4-2. Particle trajectories in the radial inflow experiment viewed in the rotating frame – i.e. the camera is co-rotating also (so that the parcel at the outer edge is at rest: $v_0 = 0$, see (1.C-5)). Left: $\Omega = 5$ rotations per minute (rpm); right: $\Omega = 10$ rpm. In the limit $\Omega \rightarrow 0$, the particle goes straight from edge to center hole, and when $\Omega >> 1$, then the particle circles around the center without “falling” into the hole.
3.5 The source of instability - parcel method:

First study baroclinic instability. Why does the zonal flow develop wave and eventually eddies (Fig.3.1-2)? The system is baroclinically unstable. Consider exchange of fluid parcel (Fig.3.5-1); if the exchange results in lowering of PE, then KE can potentially be released. For a slanted isopycnal (density surface), this is possible only for the special exchange in (iii) of Fig.3.5-1.

Fig.3.5-1 Schematized diagrams showing various scenarios when exchange of fluid parcels “A” and “B” along the blue path line either leads to increased potential energy so that the mass center of the system moves up (stable), is unchanged (neutral) or to decreased potential energy so that the mass center of the system moves down (maybe unstable). It is easy to see that only for the special path within the “wedge” formed by the isopycnal (red line) and the horizon (dashed line) in “(iii)” can the exchange possibly lead to instability (such that after the exchange the two parcels move away from each other). Such an instability is called baroclinic instability.

Consider the parcels in Fig.3.5-1 to be on a two-dimensional yz-plane and denote the positions of parcel “A” having the density of its surrounding $\rho_2$ as $(y_2,z_2)$, and “B” with density $\rho_1$ as $(y_1,z_1)$. The parcels are labelled in (iii) but the descriptions below are general applicable to parcels in any of the panels in Fig.3.5-1. We consider the potential energy (PE) states of the fluid system before and after the two parcels exchange positions, so that if the end state has a lower PE, then the parcels’ exchange may become unstable. The situation loosely resembles that of a hiker loosing PE decending a hill when he (or she) may risk becoming unstable though falling – thus converting the lost PE into kinetic energy (KE; the hiker may stabilize himself by wearing a pair of good boots with plenty of friction!). In a fluid, however, we cannot consider an individual parcel in isolation, since fluid is a continuum. The movement of one parcel affects its surrounding, so we can only compare the PE’s of different states of the fluid when parcels rearrange themselves. The simplest
rearrangement is with two parcels. Assuming that the fluid is incompressible, the PE’s (per unit volume) before and after the exchange are:

\[
\begin{align*}
\text{PE}_\text{bef} &= \rho_1 g z_1 + \rho_2 g z_2 \\
\text{PE}_\text{aft} &= \rho_2 g z_1 + \rho_1 g z_2
\end{align*}
\]

(3.5-1a)

(3.5-1b)

The assumption of incompressibility greatly simplifies the analysis as we need not worry about the pressure work by the parcel on the surrounding fluid, nor of the surrounding fluid on the parcel as it moves around. The change in PE is then:

\[
\Delta \text{PE} = \text{PE}_\text{aft} - \text{PE}_\text{bef} = g \left\{ (\rho_2 - \rho_1) z_1 - (\rho_2 - \rho_1) z_2 \right\} = -g (\rho_2 - \rho_1) (z_2 - z_1)
\]

(3.5-2)

i.e. \( b = \frac{-g \rho_2}{\rho_0} \)

(3.5-3)

is the buoyancy and \( \rho_0 \) is a constant reference density.

The expression (3.5-2) is difficult to interpret as it poses many possibilities for the sign of \( \Delta \text{PE} \). We therefore want to express it in terms of the buoyancy (i.e. density or stratification) state of the surrounding. The horizontal and vertical stratifications of the surrounding are:

\[
\begin{align*}
M^2 &= \partial b/\partial y \quad \text{and} \quad N^2 = \partial b/\partial z.
\end{align*}
\]

(3.5-4)

If these vary slowly, so they are approximately constant over \( \Delta z = (z_2 - z_1) \) and \( \Delta y = (y_2 - y_1) \), then:

\[
b_2 - b_1 \approx M^2 \Delta y + N^2 \Delta z
\]

(3.5-5)

This equation is exact when \( M^2 \) and \( N^2 \) are constant. Consider an isopycnal (i.e. a constant b-contour, the red line in Fig.3.5-1), then on that contour:

\[
\delta b = 0 = \partial b/\partial z \delta z + \partial b/\partial y \delta y;
\]

i.e. \( (\partial z/\partial y)_b = s_b = -M^2/N^2 \), \( M^2 = \partial b/\partial y \) and \( N^2 = \partial b/\partial z \)

(3.5-6)

which is the background slope of the isopycnal. The slope of line of the parcels’ exchange is:

\[
s = (z_2 - z_1)/(y_2 - y_1) = \Delta z/\Delta y.
\]

(3.5-7)

Substituting \( b_2 - b_1 \) from (3.5-5) into (3.5-2), and using (3.5-6) and (3.5-7):

\[
\Delta \text{PE} = \rho_0 N^2 (-s_b + s) \Delta y^2
\]

(3.5-8)

If there is to be an instability, \( \Delta \text{PE} < 0 \). Since \( s_b > 0 \) (see Fig.3.5-1), we then have:

\[
s_b > s > 0 \quad \text{for instability,}
\]

(3.5-9)

and the system is stable for parcel-exchange along a line with negative slope \( s < 0 \). In other words, instability can occur only if the slope of isopycnal is larger than the (positive) slope of the line of exchange of parcels – see Fig.3.5-1(iii). Maximum -\( \Delta \text{PE} \) (i.e. minimum \( \Delta \text{PE} \)) occurs when \( \partial \Delta \text{PE}/\partial s = 0 = -s_b + 2s \), or if \( s = s_b/2 \), i.e. when the exchange occurs along a line with slope half that of the isopycnal. This is the essence of baroclinic instability. The parcel method is now applied to other types of instabilities in atmosphere and ocean.
Static instability:

![Fig.3.5-2 Stability and instability of a ball on a curved surface.](image)

**Fig.3.5-2** Stability and instability of a ball on a curved surface.

An example of instability: Consider point “A” or “B” in Fig.3.5-2 – is the state of the ball stable? When perturbed, if the ball moves farther and farther from its original position, then the system is unstable; otherwise the system is stable.

For small distances \( \delta x \) from “A” (or “B”), acceleration is \( d^2 \delta x / dt^2 \); therefore along the slope,

\[
d^2 \delta x / dt^2 = -g dh/dx = -g(d^2 h/dx^2)_A \delta x\]

(Taylor expansion about “A” where \( dh/dx = 0 \))

Solution is: \( \delta x \sim \exp(\sigma \cdot t) \) where \( \sigma = \pm \sqrt{-g(d^2 h/dx^2)_A} \)

Thus: at “A” where \( (d^2 h/dx^2) < 0 \), the solution has an exponentially growing part, indicating instability; at “B” the solution is oscillatory, but stable.

When the ball is perturbed from the crest (top of hill) it moves downslope, its potential energy decreases and its kinetic energy increases – unstable. In the valley, by contrast, one must keep on supplying potential energy to push the ball up, for otherwise it drops back into the valley, rolls back and forth, then eventually comes to rest – stable.

![Fig.3.5-3 Fluid convection: schematic of shallow convection in a fluid, such as water, triggered by warming from below and/or cooling from above.](image)

**Fig.3.5-3** Fluid convection: schematic of shallow convection in a fluid, such as water, triggered by warming from below and/or cooling from above.

When a fluid such as water is heated from below (or cooled from above; Fig.3.5-3), it develops overturning motions. This seems obvious.

A thought experiment: consider the shallow, horizontally infinite fluid shown in Figure above. Let the heating be applied uniformly at the base; then the fluid should have a horizontally uniform temperature, so \( T = T(z) \) only. The fluid will be top-heavy: warmer and therefore lighter fluid below cold, dense fluid above. The situation does not contradict hydrostacy: \( \partial p/\partial z = -g \rho \). BUT,
1. Why do motions develop when we can have hydrostatic equilibrium, with no net forces?
2. Why can the motions become horizontally inhomogeneous when the heating is horizontally uniform? Answer: because the “top-heavy” state of the fluid is statically unstable.

**Fig.3.5-4** Illustration of buoyancy in an incompressible fluid such as water.

Buoyancy: Fig.3.5-4 shows a parcel of light, buoyant fluid surrounded by resting, homogeneous, heavier fluid in hydrostatic balance, so that $\frac{\partial p}{\partial z} = -g\rho$. The fluid above points A1, A, and A2 has the same density, and hence, by hydrostacy, pressures at the A points are all the same. But the pressure at B is lower than at B1 or B2 because the column of fluid above B is lighter. There is thus a pressure gradient force which drives fluid inwards toward B (blue arrows), tending to equalize the pressure along B1BB2. The pressure at B will tend to increase, forcing the light fluid upward, as indicated schematically by the red arrows.

The (upward) acceleration or buoyancy ($b$; note it is positive) of the parcel of fluid is:

$$b = -g \frac{\Delta \rho}{\rho_P}, \quad \Delta \rho = \rho_P - \rho_E \quad i.e. \text{ parcel minus environment densities.} \quad (3.5-10)$$

Condition for static instability: A fluid parcel is initially at $z_1$ in an environment whose density is $\rho(z)$ (Fig.3.5-5).

**Fig.3.5-5**

The parcel’s density $\rho_1 = \rho(z_1)$, is the same as its environment at height $z_1$. The parcel is now displaced without loosing or gaining heat (i.e. adiabatically), and also without expanding or
contracting (incompressible), a small vertical distance to \( z_2 = z_1 + \delta z \), where the new environment density is:

\[
\rho_E = \rho(z_2) \approx \rho_1 + (d\rho/dz)_E \delta z.
\]

(3.5-11)

The parcel’s buoyancy is then:

\[
b = -g(\rho_1 - \rho_E)/(\rho_1) = g(d\rho/dz)_E \delta z/\rho_1.
\]

Therefore, if

\[
(dp/dz)_E > 0 \quad \text{parcel keeps rising} \rightarrow \text{unstable}
\]

\[
(dp/dz)_E = 0 \quad \text{parcel stays} \rightarrow \text{neutrally stable}
\]

(3.5-12)

\[
(dp/dz)_E < 0 \quad \text{parcel returns} \rightarrow \text{stable}
\]

In the absence of viscous and diffusive effects, an incompressible liquid is unstable if density increases with height.

Parcel method: Consider again Fig.3.5-5. Imagine that the 2 fluid parcels (“1” and “2”) exchange their positions. Before the exchange, the PE (focusing on these 2 parcels) is:

\[
PE_{\text{before}} = g (\rho_1 z_1 + \rho_2 z_2)
\]

(3.5-13a)

After the exchange:

\[
PE_{\text{after}} = g (\rho_1 z_2 + \rho_2 z_1).
\]

(3.5-13b)

Then:

\[
\Delta PE = PE_{\text{after}} - PE_{\text{before}} = -g (\rho_2 - \rho_1) (z_2 - z_1)
\]

But from previous sub-section, \( \rho_2 - \rho_1 \approx (dp/dz)_E \delta z = (dp/dz)_E (z_2 - z_1) \), (see equation (3.5-10)).

Therefore:

\[
\Delta PE = -g(dp/dz)_E (z_2 - z_1)^2,
\]

(3.5-14)

and \( \Delta PE < 0 \) indicating instability if \( (dp/dz)_E > 0 \), \( \Delta PE = 0 \) indicating neutral stability if \( (dp/dz)_E = 0 \) and \( \Delta PE > 0 \) indicating stability if \( (dp/dz)_E < 0 \), as in the previous sub-section.

The following two figures taken from Marshall & Plumb’s book further illustrate the ideas.

**Laboratory experiment of convection**

![Laboratory experiment of convection](image)

Fig.3.5-6

Fig.3.5-6 shows a) A sketch of the **laboratory apparatus** used to study convection. A **stable stratification is set up** in a 50cm × 50 cm × 50 cm tank by slowly filling it up with water whose temperature is slowly increased with time. This is done using (1) a mixer, which mixes hot and cold
water together, and (2) a diffuser, which floats on the top of the rising water and ensures that the warming water floats on the top without generating turbulence. Using the hot and cold water supply we can achieve a temperature difference of 20°C over the depth of the tank. The temperature profile is measured and recorded using thermometers attached to the side of the tank. **Heating at the base** is supplied by a heating pad. The motion of the fluid is made visible by sprinkling a very small amount of potassium permanganate (which turns the water pink) after the stable stratification has been set up and just before turning on the heating pad. **Convection carries heat from the heating pad into the body of the fluid, distributing it over the convection layer much like convection carries heat away from the Earth’s surface.** (b) Schematic of evolving convective boundary layer heated from below. The initial linear temperature profile is $T_E$. The convection layer is mixed by convection to a uniform temperature. **Fluid parcels overshoot into the stable stratification above**, creating the inversion evident in (c). Both the temperature of the convection layer and its depth slowly increase with time.

![Image](image.png)

**Fig.3.5-7 Left**: Parcels overshoot the neutrally buoyant level, brush the stratified layer above, produce gravity waves, then sink back into the convective layer beneath  
**Right**: Temperature time series measured by five thermometers spanning the depth of the fluid at equal intervals shown on left. The lowest thermometer is close to the heating pad. We see that the ambient fluid initially has a roughly constant stratification, somewhat higher near the top than in the body of the fluid. The heating pad was switched on at $t = 150$ sec. Note how all the readings converge onto one line as the well mixed convection layer deepens over time.

**Centrifugal instability (also called inertial instability):**

Gravity provides the vertical force in static stability or instability. For homogeneous fluid, i.e. of constant density $\rho$, the system is *statically* neutrally stable. But if the fluid system spins, as for example in a typhoon or hurricane, then fluid parcel experiences centrifugal force, which acts the role of gravity but *horizontally*. Thus there may be an analogous instability condition which depends on how fast the parcel spins and what the direction of the spin is. Since the parcel is on the earth which itself is spinning, one may expect the condition to depend on the Coriolis parameter $f$. 
We apply “parcel theory” to parcels moving at velocity \( u(y) \), and the motion is independent of “\( x \)”. Thus the parcel is a long “tube.” We then consider the change of the kinetic energy (\( KE \)) of the system when two tubes, one at \( y_1 \) and the other at \( y_2 \) exchange positions. Note that since \( \rho = \rho_o \), a constant, there is no exchange of \( PE \). The exchange is unstable if \( \Delta KE < 0 \), where

\[
\Delta KE = KE_{\text{final}} - KE_{\text{initial}}
\]

so that energy is released to drive instability. As \( \partial p/\partial x = 0 \), the \( x \)-momentum equation becomes:

\[
Du/Dt - fv = Dm/Dt = 0
\]

where \( m = u - fy \) is the absolute momentum. Since \( m = \text{constant following the parcel} \):

\[
\Delta u = f\Delta y,
\]

where \( \Delta y = y_2 - y_1 \) in our case. (3.5-17)

Thus, \( (2/\rho_o) \Delta KE = (u_1 + f\Delta y)^2 + (u_2 - f\Delta y)^2 - u_1^2 - u_2^2 = 2f\Delta y(u_1 - u_2) + 2f^2\Delta y^2 \)

i.e. \( \Delta KE = \rho_o f\Delta y^2(f - \Delta u/\Delta y) \), where \( \Delta u = u_2 - u_1 \) (3.5-18)

But, \( \Delta u = \partial u/\partial y \Delta y \) (3.5-19)

So, \( \Delta KE = \rho_o \Delta y^2 f(f - \partial u/\partial y) = \rho_o \Delta y^2 f \eta_{ax} \) (3.5-20)

where \( \eta_{ax} = f - \partial u/\partial y \) (3.5-21)

is the vertical component of the absolute vorticity which in general is given by:

\[
\eta_a = (\eta_{ax}, \eta_{ay}, \eta_{az}) = (\partial w/\partial y - \partial v/\partial z, \partial u/\partial z - \partial w/\partial x, f + \partial v/\partial x - \partial u/\partial y)
\]

From (3.5-20), centrifugal instability \( \Delta KE < 0 \) occurs when:

\[ f - \partial u/\partial y < 0 \text{ for } f > 0 \]

or \[ |f| + \partial u/\partial y < 0 \text{ for } f < 0 \]

Thus for centrifugal instability to occur, it is necessary that the fluid vorticity be anticyclonic (\( \partial u/\partial y > 0 \) in NH or \( \partial u/\partial y < 0 \) in SH; note that “\( y \)” points northward). Centrifugal instability is therefore only possible if the parcel’s vorticity is opposite to the local Coriolis, i.e. only for anticyclones; it cannot occur for cyclones in both hemispheres (-\( \partial u/\partial y > 0 \) in NH or \( \partial u/\partial y > 0 \) in SH). This type of instability seldom occurs except for anticyclones near the equator.

**Symmetric (or slantwise) instability:**

We now let \( u(y,z) \) be a function of \( y \) and \( z \) so that \( \rho \) or buoyancy \( b = -g\rho/\rho_o \) is also a function of \( y \) and \( z \) (see Fig.3.5-1(iii)). For \( \Delta KE \), instead of equation (3.5-19), we now have:

\[
\Delta u = \partial u/\partial y \Delta y + \partial u/\partial z \Delta z
\]

Then, \( \Delta KE = \rho_o \Delta y^2 f(f - \partial u/\partial y - s\partial u/\partial z) \) (3.5-25)

where \( s = \Delta z/\Delta y \) is the slope of the line of exchange between the 2 parcels, defined in (3.5-7).

Thermal wind and (3.5-6) give:

---

1 In cylindrical polar coordinate – as used in an axisymmetric model of a typhoon for example – “\( m \)” would be the absolute angular momentum: \( v_\theta r + fr^2/2 \), where \( v_\theta \) is the azimuthal velocity and \( r \) = radial axis.
\[ \left( \frac{\partial \rho u}{\partial z} \right) = -M^2 = N^2 s_b \]  

Then,  
\[ \Delta KE = \rho_o \Delta \gamma \left[ f(f - \partial u/\partial y) \right] - s_b \times N^2. \]  

As \( \rho \) is not constant, the particle exchange also results in a change in PE given by (3.5-8). The total change in energy is the sum of the two:

\[ \Delta E = \rho_o \Delta \gamma \left[ f(f - \partial u/\partial y) \right] + N^2(s - s_b) \times N^2 \]

i.e.  
\[ \Delta E = \rho_o \Delta \gamma \left[ f(f - \partial u/\partial y) \right] + N^2(s - s_b)^2 - f^2/Ri; \]

where \( \text{Ri} = N^2/(\partial u/\partial z)^2 = f^2 N^2/M^4 \)

is the Richardson number. Equation (3.5-28) gives \( \Delta E \) dependent on \( s \) for a given flow environment defined by “\( u \)” and “\( \text{Ri} \).” Since \( (s - s_b)^2 \) appears as a square, and instability is characterized by \( \Delta E < 0 \), we set \( s = s_b \) to find minimum \( \Delta E \) (i.e. most negative \( \Delta E \)), and after also using (3.5-21) obtain:

\[ \Delta E_{min} = \rho_o \Delta \gamma \left[ f(f - \partial u/\partial y) \right] + N^2 \left( s - s_b \right)^2 - f^2 f \times (\eta_a/f - 1/Ri) \]

which is negative, i.e. unstable called symmetric instability, when:

\[ \text{Ri} < f/\eta_a \]

The Ertel’s potential vorticity is:

\[ Q = \eta_a \nabla b \]

i.e.  
\[ Q = \partial u/\partial z \cdot \partial b/\partial y + \eta_a \partial b/\partial z = -M^2 f + \eta_a \times N^2 = fN^2 (\eta_a/f - 1/Ri) \]

for our case of the \( yz \)-plane. Therefore \( \Delta E_{min} < 0 \) is equivalent to \( Q < 0 \) also, and the symmetric instability condition (3.5-31) corresponds to negative \( Q \). The “Q” means stratification (i.e. \( \nabla b \)) in the direction of \( \eta_a \) so that \( Q < 0 \) is actually equivalent to “static instability” condition (3.5-14) that \( \partial b/\partial z < 0 \) (recall that \( b \sim -\rho \)), since \( \eta_a \) points upward in “\( z \)” and \( \eta_a \nabla b \sim \partial b/\partial z \). Therefore, symmetric instability can be thought of as a generalized convective instability in the direction of the absolute vorticity \( \eta_a \). As in convection problem, whenever the instability occurs, i.e. whenever \( Q < 0 \), strong mixing occurs which homogenizes the “\( b \)” in the direction of \( \eta_a \) so that \( Q \) returns to \( \approx 0 \). This type of instability often occurs in the eye wall of typhoons [Emanuel 1986], along contours of constant \( \theta_e \), the saturated equivalent potential temperature (the temperature that a parcel would have if all its water vapor is condensed and the heat released is used to raise parcel’s temperature).

Table below summarizes the various types of instabilities.

<table>
<thead>
<tr>
<th>Instability</th>
<th>Condition</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Static</td>
<td>((dp/dz)_E &gt; 0 ) or ((db/dz)_E &lt; 0 )</td>
<td>3.5-14</td>
</tr>
<tr>
<td>Centrifugal</td>
<td>( f - \partial u/\partial y &lt; 0 ) (for ( f &gt; 0 )) [ \text{or} ] (</td>
<td>f</td>
</tr>
<tr>
<td>Symmetric</td>
<td>( \text{Ri} &lt; f/\eta_a ) (i.e. ( \eta_a \nabla b &lt; 0 ))</td>
<td>(3.5-31)</td>
</tr>
<tr>
<td>Baroclinic</td>
<td>( s_b &gt; s &gt; 0 )</td>
<td>(3.5-9)</td>
</tr>
</tbody>
</table>
What about if background flow has horizontal shear $u \rightarrow u + U(y)$?

\[
\Delta u = f\Delta y - \Delta U, \quad \text{(3.5-17)}
\]

\[
f' = f - \frac{dU}{dy}
\]

Thus, \((2\rho_o)\Delta KE\)

\[
= (u_1 + f\Delta y - U_1 - dU/dy\Delta y)^2 + (u_2 - f\Delta y - U_2 + dU/dy\Delta y)^2 - (U_1 + u_1)^2 - (U_2 + u_2)^2
\]

\[
= [(u_1 - U_1)^2 + 2(u_1 - U_1)f\Delta y + (f\Delta y)^2] + [(u_2 - U_2)^2 - 2(u_2 - U_2)f\Delta y + (f\Delta y)^2] - (U_1 + u_1)^2 - (U_2 + u_2)^2
\]

\[
= -4u_1U_1 - 4u_2U_2 + 2(f\Delta y)^2 + 2f'\Delta y(u_1 - u_2 + dU/dy\Delta y)
\]

\[
= (u_1 + f\Delta y - U_1 - dU/dy\Delta y)^2 + (u_2 - f\Delta y - U_1)^2 - (U_1 + u_1)^2 - (U_2 + u_2)^2
\]

\[
U_2 = U_1 + dU/dy\Delta y
\]

\[
= 2f\Delta y(u_1 - u_2) + 2f^2\Delta y^2
\]

i.e. \(\Delta KE = \rho_o f^2\Delta y^2(f - \Delta u/\Delta y)\), where \(\Delta u = u_2 - u_1\) \(\text{(3.5-18)}\)

But, \(\Delta u = \partial u/\partial y\Delta y\) \(\text{(3.5-19)}\)

So, \(\Delta KE = \rho_o f^2\Delta y^2 f(f - \partial u/\partial y) = \rho_o f^2\Delta y^2 f\eta_{az}\) \(\text{(3.5-20)}\)

where \(\eta_{az} = f - \partial u/\partial y\) \(\text{(3.5-21)}\)

is the vertical component of the absolute vorticity which in general is given by:

\[
\eta_a = (\eta_{ax}, \eta_{ay}, \eta_{az}) = (\partial w/\partial y - \partial v/\partial z, \partial u/\partial z - \partial w/\partial x, f + \partial v/\partial x - \partial u/\partial y)
\]

\(\text{(3.5-22)}\)

From \(3.5-20\), centrifugal instability \(\Delta KE < 0\) occurs when:

\[
f - \partial u/\partial y < 0 \quad \text{for} \quad f > 0 \quad \text{(3.5-23a)}
\]

or \(|f| + \partial u/\partial y < 0 \quad \text{for} \quad f < 0 \quad \text{(3.5-23b)}\)
3.6 Heuristic derivation of the quasigeostrophic (QG) potential vorticity (PV) equation for a stratified fluid

To study baroclinic instability in more details, we develop the PV equation similar to (2.8-7):
\[ \frac{\partial}{\partial t} (\nabla^2 \psi - \psi / R^2) + J(\psi, \nabla^2 \psi - \psi / R^2) + \beta \frac{\partial \psi}{\partial x} = 0 \]

which was for barotropic flow (on \( \beta \)-plane with zero forcing), but now we take account also of stratification. A heuristic approach is given below – for a rigorous derivation see Pedlosky (1979).

The inviscid x, y & z-momentum and continuity equations are:

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - \nu v = -\rho_i^{-1} (\partial p/\partial x) \]  
\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + f u = -\rho_i^{-1} (\partial p/\partial y) \]  
\[ \frac{\partial p}{\partial z} = -g \rho \]  
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]

Also, a beta-plane is assumed such that:
\[ f = f_o + \beta_o y \]  

where \( f_o \) is the Coriolis parameter at \( y=0 \) (around where later we will develop the instability analysis), \( \beta_o = df/dy \), and it is also assumed that \( |\beta_o y| << |f_o| \). For \( \beta_o \approx 10^{-11} \text{ m}^{-1} \text{ s}^{-1} \), and \( |f_o| \approx 5 \times 10^{-5} \text{ s}^{-1} \) or larger (poleward of 20°N/S), we restrict \( |y| \approx 1000 \text{ km} \) or less.

The symbols have the usual meanings, and \( \rho_i = \text{ reference density which is a function of } z \) only. The basic QG-approximation idea is that the motions are close to being geostrophic, so we will use the geostrophic velocity assuming constant \( f \approx f_o \) (which is ‘zeroth-order’) to evaluate the “difficult” terms which in (3.6-1a,b) are the non-linear and “\( \beta_o \)” terms.

To the zeroth-order approximation, the motions are nearly geostrophic, so that:

\[ -f_o v_o = -\rho_i^{-1} (\partial p_o/\partial x) \]  
\[ +f_o u_o = -\rho_i^{-1} (\partial p_o/\partial y) \]  

so that
\[ \partial u_o/\partial x + \partial v_o/\partial y = -\partial w_o/\partial z \approx 0 \]
Therefore, \( w_o \approx 0 \) since it is \( \approx 0 \) at the surface. In any case, the zeroth-order or geostrophic approximation of \( w = w_o \) is small compared to \( |u_o| \) or \( |v_o| \).

Substitute the \((u_o, v_o, w_o=0)\) into the non-linear and beta parts of (3.6-1), we then obtain the following equations for the next-order term with subscripts “1” \((u_1, v_1, p_1)\):

\[
\begin{align*}
\frac{d}{dt}\{u_o\} - f_o v_1 - \beta_o y v_0 &= -\rho^{-1} \frac{\partial p_1}{\partial x} \\
\frac{d}{dt}\{v_o\} + f_o u_1 + \beta_o y u_0 &= -\rho^{-1} \frac{\partial p_1}{\partial y}
\end{align*}
\]

where

\[
\frac{d}{dt} = \partial/\partial t + u_o \partial/\partial x + v_o \partial/\partial y.
\]

Taking the \((z\text{-component of the})\) curl of (3.6-5) to eliminate the pressure, and using the first of (3.6-4) that \( \partial u_o/\partial x + \partial v_o/\partial y \approx 0 \), we obtain:

\[
\begin{align*}
\frac{d}{dt}\{\zeta_o\} + \beta_o y v_0 + f_o \nabla_H \cdot \mathbf{u}_1 &= 0, \text{ where } \nabla_H = i \partial/\partial x + j \partial/\partial y \text{ and } \mathbf{u}_1=(u_1,v_1); \\
\text{or,} \\
\frac{d}{dt}\{\zeta_o + \beta_o y\} &= f_o \partial w_1/\partial z \quad (3.6-7)
\end{align*}
\]

after using (3.6-6) and (3.6-1d), i.e. \( \partial u_1/\partial x + \partial v_1/\partial y + \partial w_1/\partial z = 0 \).

For \( \rho = \text{constant} \), by vertically integrating (from \( z=-H \) to \( z=0 \)) equation (3.6-7) we recover the QGPV equation for a homogeneous fluid, which we studied previously in section 2-1 (see equation (2-10)). For stratified fluid, we need to evaluate \( \partial w_1/\partial z \) using the density (or buoyancy \( b = -\partial \rho/\partial z \)) equation [see section 11.1.1 of Marshall and Plumb, 2008] which in the absence of heat/salt sources and sinks on the RHS is:

\[
\partial \rho/\partial t + u \partial \rho/\partial x + v \partial \rho/\partial y + w \partial \rho/\partial z = 0 \quad (3.6-8)
\]

The density is given by:

\[
\rho = \rho(z) + \rho'(x,y,z,t) \quad (3.6-9)
\]

and the hydrostatic equation is then assumed also for the perturbation density \( \rho' \) (and pressure \( p_o \)):

\[\text{Typically, } w = \text{Ekman pumping near the surface } \approx \nabla \times \mathbf{\tau}/(f \rho_o) \approx 0.1(\text{N/m}^2)/[10^4(\text{m}).10^4(\text{s}^{-1}).10^3(\text{kg/m}^3)] \approx 10^{-4} \text{ m/s} \approx 10 \text{ m/day}, \text{ for a strong wind stress curl of } 0.1 \text{ N/m}^2 \text{ over a distance of } 10 \text{ km. Thus comparing to typical } |u_o| \approx 5 \times 10^{-2} \text{ m/s, the “w” is indeed small. In the open ocean, the “w” is typically smaller than 10 m/day.}\]
\( \partial p_o/\partial z = -g \rho' \) (also of course \( \partial p_o/\partial z = -g \rho_i \) ) \hfill \text{(3.6-10a)}

or 

\( (\partial p_o/\partial z) / \rho_i = -g \rho' / \rho_i = b' \) \hfill \text{(3.6-10b)}

Using the QG-approximation on (3.6-8), we get:

\[
d_o/dt \{ \rho' \} + w_1 (d \rho_i / dz) = 0, \tag{3.6-11a}
\]

or multiplying by \(-g/\rho_i\), we get \( d_o/dt \{-g \rho'/\rho_i\} + w_1 (-g d \rho_i / dz) / \rho_i = 0\), i.e.

\[
d_o/dt \{ b' \} + w_1 N^2_i(z) = 0 \tag{3.6-11b}
\]

where \( N^2(z) = (-g d \rho_i / dz) / \rho_i \) is the squared buoyancy (or Brunt-Vaisala) frequency. We see that (in the absence of sources and sinks) the vertical velocity \( w_1 \) is related to the isopycnal movement due to the geostrophic flow:

\[
w_1 = -d_o/dt \{ b'/N^2_i(z) \}, \quad \text{or also} \quad d_o/dt \{ \rho'/\{-d \rho_i / dz \} \}. \tag{3.6-11c}
\]

The last form shows, since \(-d \rho_i / dz > 0\), that deepening isopycnal (i.e. \( d_o \{ \rho' \}/d t < 0 \)) corresponds to downwelling, and shallowing isopycnal corresponds to upwelling.

Using the hydrostatic equation (3.6-10b): \([\partial p_o/\partial z]/\rho_i = b'\), so that (3.6-11c) becomes:

\[
w_1 = -d_o/dt \{ (\partial p_o/\partial z) / (\rho_i N^2_i(z)) \}. \tag{3.6-11d}
\]

We now use \( w_1 \) in (3.6-7):

\[
d_o/dt \{ \zeta_o + \beta_o y \} = f_o \partial w_1/\partial z = -f_o d_o/dt \{ \partial [b'/N^2_i(z)] / \partial z \}
\]

or upon using the hydrostatic equation (3.6-10b):

\[
d_o/dt \{ \zeta_o + \beta_o y \} = -f_o d_o/dt \{ \partial [(\partial p_o/\partial z) / (\rho_i N^2_i)] / \partial z \}
\]

i.e.,

\[
d_o/dt \{ \zeta_o + \beta_o y + f_o \partial [(\partial p_o/\partial z) / (\rho_i N^2_i)] / \partial z \} = 0 \tag{3.6-12a}
\]

or,

\[
d_o/dt \{ (\nabla^2 p_o) / (f_o \rho_i) + \beta_o y + f_o \partial [(\partial p_o/\partial z) / (\rho_i N^2_i)] / \partial z \} = 0 \tag{3.6-12b}
\]

since from (3.6-3a,b), we have for the geostrophic vorticity:

\[
\zeta_o = \partial v_o / \partial x - \partial u_o / \partial y = \nabla^2 p_o / (f_o \rho_i) \tag{3.6-13}
\]
For the ocean (mpiPOM exp002), $\rho_r$ can be assumed to be $\approx$ constant in (3.6-12b), so that equation becomes:

$$
\frac{d\psi}{dt}\{\nabla^2(\rho_o/f_o\rho_r) + \beta_o y + \partial[(\partial(\partial(\psi/f_o\rho_r)/\partial z)(f_o^2/N^2))] / \partial z\} = 0
$$

(3.6-14)

From the geostrophic relation (3.6-3), the $\rho_o/(f_o\rho_r)$ is equivalent to geostrophic stream function:

$$
\psi = \frac{\rho_o}{f_o\rho_r}
$$

(3.6-15)

so that (3.6-14) is usually expressed in terms of $\psi$:

$$
\frac{d\psi}{dt}\{\nabla^2\psi + \beta_o y + \partial[(\partial\psi/\partial z)(f_o^2/N^2)] / \partial z\} = 0
$$

(3.6-16a)

The quantity inside the {..} is the QGPV for a stratified fluid:

$$
Q = \nabla^2\psi + \beta_o y + \partial[(\partial\psi/\partial z)(f_o^2/N^2)] / \partial z
$$

(3.6-16b)

Comparing with the barotropic QGPV:

Set $\beta_T = \beta_o$ in (2.8-7), we can write that equation as:

$$
\partial(\nabla^2\psi + \beta_o y - \psi/R_o^2) / \partial t + J(\psi, \nabla^2\psi + \beta_o y - \psi/R_o^2) = \nabla x[(\tau^0 - \tau_b)/H].
$$

(3.6-17)

Note that the last term in (3.6-16a) is (the minus sign is because PV decreases as $\delta z$ increases, and with a single layer, the decreased PV is accomplished through a decreased $\eta$) $\sim \delta \psi < 0$:

$$
\partial[(\partial\psi/\partial z)(f_o^2/N^2)] / \partial z \sim -\psi. f_o^2 / \{[g|\Delta \rho|/(\rho_o\Delta z)].\Delta z^2\}
$$

$$
= -\psi. f_o^2 / (g|\Delta \rho|\Delta z/\rho_o)
$$

$$
= -\psi. f_o^2 / C_i^2
$$

$$
= -\psi/R_o^2,
$$

(3.6-18)

where $R_o = C_i^2/f_o^2$ is called the baroclinic Rossby radius deformation.

Then (3.6-16a) can be written as, adding also curls of wind and bottom stresses on RHS:

$$
\partial(\nabla^2\psi + \beta_o y - \psi/R_o^2) / \partial t + J(\psi, \nabla^2\psi + \beta_o y - \psi/R_o^2) = \nabla x[(\tau^0 - \tau_b)/H].
$$

(3.6-19)

which is equivalent to (2.8-7). For $R_o \sim \infty$, we have unstratified, barotropic motions. For finite $R_o$, the equation reduces to the so-called reduced-gravity model.

**Numerical Solution to the 1-layer QGPV model (3.6-19)**

Approximating the bottom friction by:

$$
\tau_b = r. u_o
$$

(3.6-20)

where $u_o$ is the geostrophic velocity $k \times u_o = -(g/f_o)\nabla \eta = -\nabla \psi$, $\psi = (g/f_o)\eta$ (see section 2.8), then:
\[
\n\n\n\n\n\n\n\n\n
\n\n\n\n\n\n\n\n\n\n
Then the 1-layer QGPV equation to solve is:
\[
\frac{\partial (\nabla^2 \psi + \beta_o y - \psi/R_o^2)}{\partial t} + J(\psi, \nabla^2 \psi + \beta_o y - \psi/R_o^2) = \nabla \times \tau^\theta/H - (r/H)\nabla^2 \psi.
\]
(3.6-22)

Let \( \chi = \partial \psi/\partial t \)
(3.6-23)

Then (3.6-22) becomes:
\[
\nabla^2 \chi - \chi/R_o^2 = F(x, t)
\]
(3.6-24a)

where
\[
F(x, t) = J(\psi, \nabla^2 \psi + \beta_o y - \psi/R_o^2) + \nabla \times \tau^\theta/H - (r/H)\nabla^2 \psi
\]
(3.6-24b)

contains all but the time derivative of \( \psi \). We can solve the Poisson-type equation (3.6-24a) quite easily. Once \( \chi \) is obtained, the \( \psi \) can then be found by time-stepping integration of (3.6-23).

\[
\begin{align*}
x_i &= i\Delta x, \ i = 0, 1, 2, \ldots, M \\
y_j &= j\Delta y, \ j = 0, 1, 2, \ldots, N
\end{align*}
\]
(3.6-25)

where \( \Delta x = L_x/M \) and \( \Delta y = L_y/N \). To simplify, let \( \Delta x = \Delta y = d \). Then let:
\[
\Psi_{i,j} = \psi(x_i, y_j) \quad \text{and} \quad \chi_{i,j} = \chi(x_i, y_j)
\]
(3.6-26)

Fig.3.6-1

All the terms in “\( F \)” can be replaced by finite-differences, and then from (3.6-24a):
\[
\left[\chi_{i+1,j} - (2+R_o^2)\chi_{i,j} + \chi_{i-1,j}\right]/d^2 + \left[\Psi_{i,j+1} - (2+R_o^2)\Psi_{i,j} + \Psi_{i,j-1}\right]/d^2 = F_{i,j}
\]
(3.6-27)

Then (3.6-23) can be used in a time-stepping scheme (e.g. as in mpiPOM) to solve for a new \( \psi \).

Outline:

1. Guess \( \Psi_{i,j} \); 2. Calculate \( F_{i,j} \); 3. Solve Poisson-type equation for \( \chi_{i,j} \); 4. Find new \( \Psi_{i,j} \); 5. Goto step2 \( \rightarrow \) until changes in old and new \( \chi \) and \( \Psi \) are very small.
Solving the linearized QGPV equation:

In steady state, letting the RHS involving wind stress curl to be $S_o$ and after non-dimensionalization, equation (3.6-22) becomes:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + a^1 \frac{\partial \psi}{\partial x} = S_o(x,y), \quad (3.6-28)^n$$

We solve this by posing it as a time-dependent problem:

$$\frac{\partial \psi}{\partial t} = \nabla^2 \psi + a^1 \frac{\partial \psi}{\partial x} - S_o(x,y) \quad (3.6-29)^n$$

where now the “$t$” is the real time. We solve (3.6-27) by time-stepping until a steady state is reached so that $\partial \psi/\partial t = 0$, so that the final solution “$\psi_{\text{steady}}$” is the solution we want for the original time-independent equation (3.6-26). An “ADI” (Alternating Direction Implicit) scheme is used:

\[\begin{align*}
(\psi_{n+1}^{n+1/2} - \psi^n)/(\Delta t/2) &= a^1 \frac{\partial \psi_{n+1}^{n+1/2}}{\partial x} + \frac{\partial \psi_{n+1/2}}{\partial x} + \frac{\partial \psi_{n+1/2}}{\partial y} + \frac{\partial \psi_{n+1/2}}{\partial y} - S_o \\
(\psi_{n+1}^{n+1/2} - \psi_{n}^{n+1/2})/(\Delta t/2) &= [a^1 \frac{\partial \psi_{n+1/2}}{\partial x} + \frac{\partial \psi_{n+1/2}}{\partial y} + \frac{\partial \psi_{n+1/2}}{\partial y}] - S_o)
\end{align*}\]

where \(\frac{\partial \psi}{\partial x} = (\psi_{i+1,j} - \psi_{i,j})/(2\Delta x)\), \(\frac{\partial \psi}{\partial y} = (\psi_{i,j+1} - \psi_{i,j})/(2\Delta y)\), and \(\frac{\partial \psi}{\partial y} = (\psi_{i,j+1} - \psi_{i,j-1})/(2\Delta y)\).

Adding (3.6-28a)\(^n\) and (3.6-28b)\(^n\), we get:

\[\psi_{n+1}^{n+1/2} - \psi^n)/\Delta t = [a^1 \frac{\partial \psi_{n+1/2}}{\partial x} + \frac{\partial \psi_{n+1/2}}{\partial y} + \frac{\partial \psi_{n+1/2}}{\partial y}] - S_o \quad (3.6-31)^n\]

which is seen as a “centered-space” and “centered-time” differencing of (3.6-27)\(^n\).

Given $\psi^{(0)}$ (for $n=0$), the term inside the “[...]” on the RHS of (3.6-28a)\(^n\) is known, and that equation can be solved using the tridiagonal solver. Then after obtaining the $\psi^{(1/2)}$, the terms inside the “[...]” on the RHS of (3.6-28b)\(^n\) are known and that equation can be solved for $\psi^{(1)}$. The process is then repeated (in a “do loop”) to time-step with $n = 1$ to solve for $\psi^{(1+1/2)}$ and $\psi^{(2)}$, … etc until:

\[\text{Max}_{i,j}(\|\psi^{(N+1)} - \psi^{(N)}\|)/\text{Max}_{i,j}(\|\psi^{(N+1)}\|) < \varepsilon, \text{ for small } \varepsilon = 10^{-3} \text{ say.} \quad (3.6-32)^n\]

Linearized steady-state form of the 1-layer QGPV: response of atmosphere to an SST front:

The response of wind to an SST front is modeled in Chapters 1.C (equations (1.C-70 through 95) and 1.D of Oey-IntroOceanDyn.*. The 1-layer QGPV equation in non-dimensionalized form is (1.D-33b):

$$d\xi/dt + \beta_v v = r_v \cdot (\nabla \times \mathbf{r}^\theta - (\xi - \nabla^2 T \cdot \beta_v))/2, \quad (1.D-33b)^n$$

The subscript “$n$” in the equation (..)\(^n\) is used as a reminder that this is “nondimensionalized.” Rewritten in the form of (3.6-19) including also the “$\psi/R_o$” term, but non-dimensionalized:
\[
\partial (\nabla^2 \psi + \beta_c \psi / R_n^2) / \partial t + J(\psi, \nabla^2 \psi + \beta_c \psi / R_n^2) = r_e \{ \nabla \times \tau^o - (\nabla^2 \psi - \nabla^2 T, \beta_c') \} / 2 \quad (3.6-26)^o
\]

where \( R_n^2 = R_o^2 / L^2 \); (nondimensional Rossby radius);
\( \beta_c = \beta_o L^2 / U \approx O(1); \) see (1.D-13);
\( r_e = E_v^{1/\varepsilon}; \)
\( E_v^{1/\varepsilon} = \delta_E / D, \) the Ekman number;
\( \delta_E = \) boundary layer height;
\( D = \) layer depth; (e.g. the troposphere);
\( \varepsilon = U/(f_0 L), \) the Rossby number;
\( \beta_c' = [\alpha' T_o, \varepsilon / Fr], G(k); \) (from (1.C-75b) & (1.D-86b))
\( G(k) = [1 - k + k^2 / 2] / [k(1 + k^4 / 4)]; \)
\( \alpha' \approx - (\partial \rho / \partial \theta) / \rho_o \approx 0, \) (ideal gas law; \( \theta_o \approx 300 \) K);
\( Fr = U^2 / (g \delta_E) \) is the Froude number based on \( \delta_E; \)
\( U, L \& T_o \sim (\theta_o) \) are velocity, length & temperature scales; and

\( k > 0 \) is a dimensionless parameter that measures how deep the ocean’s influence \( (T) \) penetrates into the troposphere. For \( k >> 1, \) the ocean’s influence is confined near the sea-surface. For smaller \( k, \) the ABL is more mixed, and the ocean’s influence extends to greater heights.

The dimensional form of (3.6-26)^o (Homework 3.6.1) is:

\[
\partial (q) / \partial t + J(\psi, q) = D^1 \nabla \times \tau^o - a \nabla^2 \psi + b \nabla^2 T \quad (3.6-27)
\]

where all variables are now dimensional, \( q = \nabla^2 \psi + \beta_o y - \psi / R_o^2 \) (the total potential vorticity), \( \psi \) is the geostrophic stream function, \( \tau^o = \) stress/\( \rho \) applied at the top of the layer (\( \approx 0 \) in the present atmospheric case), \( a = (f_o / 2)(\delta_E / D), b = (f_o / 2)(\delta_E / D)^2 \alpha'R_o^2 G(k), G(k) = [1 - k + k^2 / 2] / [k(1 + k^4 / 4)], R_o = (gD)^{1/2} / f_o \) is the Rossby radius, \( g = \) acceleration due to gravity, and \( f_o = \) Coriolis parameter.

Numerical values are: \( k \approx 1, f_o = 6.83 \times 10^{-5} \) s\(^{-1} \) (at 28°N), \( \delta_E / D = 0.1, \) and \( \alpha' \approx 1/300K. \)

Linearize (3.6-27), and assume steady state \( (\partial / \partial t = 0); \)

\[-\beta_o \partial \psi / \partial x = a \nabla^2 \psi - S_o(x,y) \quad (3.6-28a)\]

where \( S_o(x,y) = b \nabla^2 T + D^1 \nabla \times \tau^o \) is the source term. (3.6-28b)

In above, we have assumed that the \( x \) and \( y \)-scales are the same \( \approx L. \) But suppose that the \( x \)-scale is much larger: \( L_x >> L_y, \) then the Laplacian is:

\[
\nabla^2 \approx \partial^2 / \partial x^2 + \partial^2 / \partial y^2 \approx \partial^2 / \partial y^2; \quad O(L_x^{-2}) \quad O(L_y^{-2})
\]
Then (3.6-28) becomes:

\[-\beta_o \partial \psi / \partial x = a_0 \partial^2 \psi / \partial y^2 - S_o(x, y)\]  (3.6-29)

This equation can be simplified by non-dimensionalizing it using the following scales:

\[\psi_o = g \delta_E / f_o; \quad L \text{ and } T_o \quad \text{(NOTE: } L \text{ is really } \sim L_y)\]  (3.6-30)

Therefore, we define the non-dimensional variables:

\[\psi_n = \psi / \psi_o; \quad (x_n, y_n) = (x, y) / L; \quad T_n = T / T_o;\]

\[a_n = a / (\beta_o L) = (f_o / 2)(\delta_E / D) / (\beta_o L) \approx 0.25 \text{ or } 2.5\]
using \(f_o = 10^{-4} \text{s}^{-1}, \delta_E / D = 0.1, \beta_o = 2 \times 10^{-11} \text{m}^{-1} \text{s}^{-1} \& L = 1000 \text{ or } 100 \text{ km}.

\[b_n = \frac{(bT_o) / (\beta_o L \psi_o)}{(f_o / 2)(\delta_E / D) / (\beta_o L)} = \frac{(f_o^2 / 2)(\delta_E / D)G(k) = a_oG(k),}{(\beta_o L)^2} \text{(after using } \alpha^p T_o \approx 1 \text{ and } R_o^2 = gD/f_o^2),\]

The equation (3.6-29) becomes non-dimensionalized; after dropping \(\nabla \times \tau^o (=0 \text{ for atmosphere})\):

\[-\partial \psi_n / \partial x_n = a_n \{ \partial^2 \psi_n / \partial y_n^2 - G(k) \nabla_n^2 T_n \}\]  (3.6-31)

Let \(t_n = -x_n a_n = -x(f_o / 2)(\delta_E / D) / (\beta_o L^2)\) \hspace{0.5cm} (3.6-32a)

then \(\partial \psi_n / \partial t_n = \partial^2 \psi_n / \partial y_n^2 - S_{on}(t_n, y_n)\), \hspace{0.5cm} (3.6-32b)

\[S_{on} = G(k)(a_n^2 \partial^2 \psi_n / \partial t_n^2 + \partial^2 / \partial y_n^2)T_n.\]  (3.6-32c)

In summary, we have converted the QGPV equation into a heat diffusion equation. Equation (3.6-31) says that for a well-posed problem, only left-half of the x-space is allowed: \(x \leq 0\).

Thus we solve the following non-dimensionalized heat diffusion equation, with a source:

\[\partial \psi / \partial t = \partial^2 \psi / \partial y^2 - S_o\]  (3.6-33a)

\[S_o = G(k)(a^2 \partial^2 T / \partial t^2 + \partial^2 T / \partial y^2)\]  (3.6-33b)

and \[G(k) = [1-k+k^2/2]/[k(1+k^4/4)],\]  (3.6-33c)

where for clarity the subscripts “n” on variables are omitted, in a semi-infinite channel:

\[-y_L \leq y \leq y_L \quad \text{and} \quad 0 \leq t < \infty \quad \text{(or } -\infty < x \leq 0);\]  (3.6-34)

The boundary and initial conditions are:

\[\psi(t, -y_L) = 0 \text{ (at } y = -y_L) \quad \text{and} \quad \psi(t, y_L) = 0 \text{ (at } y = y_L).\]  (3.6-35)

\[\psi(0, y) = 0.\]  (3.6-36)
Using the “heat” analogy, $\psi$ is like the temperature in a metal rod of length $y_L$. The above initial and boundary conditions suggest that the temperature will remain zero. But, we have the source $S_o$, and temperature will therefore be not zero. To specify source, we need “$T$”:

\[
T(y) = -\Delta T \tanh(y/d), 
\]

(3.6-37a)

\[
T_y = -\Delta T/d \cosh^2(y/d) = -\Delta T/d \operatorname{sech}^2(y/d) 
\]

(3.6-37b)

\[
T_{yy} = (2\Delta T/d^2) \tanh(y/d) / \cosh^2(y/d) 
\]

(3.6-37c)

**Numerical values:**

$\Delta T = 0.02 (= 6/T_o$ for $T_o = 300K)$, $d = 1 (= 100\text{km}/L)$; $y_L = 10$ (units);

$k = 1$; $a = 1.7075 (= (f_o/2)(\delta_E/D)/(\beta_o L))$;

$f_o = 6.83 \times 10^{-5} \text{s}^{-1}$ (at 28°N), $\delta_E/D = 0.1$; $\beta_o=2\times10^{-11} \text{m}^{-1}\text{s}^{-1}$ & $L = 100$ km.

**Homework 3.6.2:** Integrate (3.6-33) with (3.6-34,35&36) from $t = 0$, to $t = 20$ using implicit method – tridiagonal solver (see equation (1.1-20) and (1.1-21) but now add a source term). Then:

1. Plot $\psi$-contours on $ty$-plane: $\psi(t,y)$ for $t=0\rightarrow 20$ and $y=-10\rightarrow +10$;
2. Then change $t$ into $x$ using $x = -t/a$, and replot $\psi(x,y)$-contours on $xy$-plane for $x=-12\rightarrow 0$ (corresponding to $t=20\leftarrow 0$) and the same $y$ ($-10\rightarrow +10$) (see Fig.3.6-2, upper panel);
3. Calculate $u = -\partial \psi / \partial y$ and $v = \partial \psi / \partial x$, and replot (2) with vectors superimposed.

**Homework 3.6.3:** Write a general “subroutine source” for $S_o$ by approximating $\partial^2 T/\partial t^2$ & $\partial^2 T/\partial y^2$ by $2^{nd}$ order finite-differences: input is “$T(t,y)$” and output is “$S_o$”. Repeat (1)-(3) of HW 3.6.2;

**Homework 3.6.4:** Use “subroutine source” and new $T(t,y) = -\Delta T \tanh(y/d).\exp\left[-(t-t_o)/\tau_t^2\right]$, $\tau_t = 4$, $t_o=2$, repeat (1)-(3) of HW 3.6.2;

**Homework 3.6.5:** Rotate the $T$-front, so that direction of warm-to-cold SST points northwestward instead of northward (Fig.3.6-2, lower). To do that, in the formula for $S_o$, replace $(t,y)$ by $(t’,y’)$: $t’ = t \cos(\gamma) - a.y \sin(\gamma)$ and $y’ = (t/a) \sin(\gamma) + y \cos(\gamma)$, where $\gamma = \pi/4$ (i.e. 45°). This is only approximate however; the exact way is to change the equation (3.6-33a)

(a) Use the “$T$” given by (3.6-37a) which is now $T(t’,y’) = -\Delta T \tanh\left\{ (t/a) \sin(\gamma) + y \cos(\gamma) / \tau_t \right\}$,

plot the $T$-contours to make sure that the front is tilted correctly;

(b) Use “subroutine source”, and the “$T$” from (a) above, repeat (1)-(3) of HW 3.6.2;
Fig. 3.6-2. Upper panel: after \( \psi(t,y) \) is obtained (ty-plane), then \( \psi(x,y) \) (xy-plane) can be obtained by using \( x = -t/a \), and replotting. The example shown uses \( a = 1.7075 \). Lower panel: the SST front is rotated with respect to true north.

**Homework 3.6.6:**
Rotate the front like in the previous HW, but now using the “T” of HW 3.6.4, i.e.:

\[
T(t',y') = -\Delta T \tanh(y'/d) \exp[-(t'-t_0)^2/r_t^2],
\]

with the same \( r_t = 4 \) & \( t_0 = 2 \);
Plot again the T-contours to make sure that the front is tilted correctly; then use “subroutine source”, and the “T” from (a) above, repeat (1)- (3) of HW 3.6.2.

**Homework 3.6.7:**
Instead of (1.1-21), repeat HW 3.6.3 (Soon&SM), 3.6.4 (SM&YY), 3.6.5 (Jia&RG) & 3.6.6 (YY&RG) using the more accurate trapezoidal rule:

\[
(\psi_j^{n+1} - \psi_j^n)/\Delta t = \left( (\psi_{j+1}^{n+1} - 2\psi_j^{n+1} + \psi_{j-1}^{n+1}) + (\psi_{j+1}^n - 2\psi_j^n + \psi_{j-1}^n) \right)/(2\Delta y^2) - [S_j^{n+1} + S_j^n]/2.
\]

In this new scheme, we are averaging the RHS of (3.6-33a)\(^n\) using both “n+1” and “n” time steps.

**An Exact Way to Account for the Orientation of the SST Front:**
Let \((x', y')\) be the new axes rotated at an anticlockwise angle \(\gamma\) from the original \((x, y)\) axes (which point east and north; Fig. 3.6-2 bottom panel). The SST contours (the front) are aligned with the \(x'\)-axis and positive \(y'\) points in the direction of decreasing SST. Then,

\[
x' = x.C_\gamma + y.S_\gamma \quad \text{and} \quad y' = -x.S_\gamma + y.C_\gamma
\]  

(3.6-38)*

where \(C_\gamma = \cos(\gamma)\) and \(S_\gamma = \sin(\gamma)\). By chain rule, \(\partial/\partial x = \partial/\partial x'. \partial x'/\partial x + \partial/\partial y'. \partial y'/\partial x\), etc., we obtain:

\[
\partial/\partial x = C_\gamma. \partial/\partial x' - S_\gamma. \partial/\partial y' \quad \text{and} \quad \partial/\partial y = S_\gamma. \partial/\partial x' + C_\gamma. \partial/\partial y'
\]  

(3.6-39a)*

and

\[
\nabla^2 \equiv \partial^2/\partial x'^2 + \partial^2/\partial y'^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 \equiv \nabla^2
\]  

(3.6-39b)*

Equation (3.6-28a) then gives:

\[
-C_\gamma. \partial \psi/\partial x' + S_\gamma. \partial \psi/\partial y' = (a/\beta_o). \nabla^2 \psi - S'_o(x', y')
\]  

(3.6-40)*

where the “**” on equation# is a reminder that all variables in this equation are dimensional – as are also those in (3.6-28a). We can now go through exactly the same argument that the along-frontal scale is much larger than the across-frontal scale, i.e. \(L_x \gg L_y\), so that:

\[
-C_\gamma. \partial \psi/\partial x' + S_\gamma. \partial \psi/\partial y' = (a/\beta_o). \partial^2 \psi/\partial y'^2 - S'_o(x', y')
\]  

(3.6-41)*

which replaces (3.6-29). The non-dimensionalized equation is then:

\[
-C_\gamma. \partial \psi_u/\partial x'_n + S_\gamma. \partial \psi_u/\partial y'_n = a_n. \{ \partial^2 \psi_u/\partial y'^2 - G(k) \nabla^2 \psi_u \}
\]  

(3.6-42)

which replaces (3.6-31) and has on the LHS an “advective” term “\(S_\gamma. \partial \psi_u/\partial y'_n\)”. Finally, instead of (3.6-33)*, we have the following non-dimensionalized advection-diffusion equation with a source:

\[
C_\gamma. \partial \psi/\partial t = -S_o. \partial \psi/\partial y + \partial^2 \psi/\partial y^2 - S_o
\]  

(3.6-43a)*

\[
S_o = G(k). (a^2 \partial^2 \psi/\partial t^2 + \partial^2 \psi/\partial y^2)
\]  

(3.6-43b)*

\[
G(k) = [1-k+k^2]/[k(1+k^2/4)], \quad C_\gamma = \cos(\gamma) \quad \text{and} \quad S_{o\gamma} = \sin(\gamma)/a.
\]  

(3.6-43c)*

For convenience the primes on \(x'\) and \(y'\) are omitted.

For Homeworks 3.6.8-3.6.11 below, you need to download RG’s codes & *pdf notes from:

ftp://profs.princeton.edu/leol/lecture-notes/OceanAtmosModeling/Codes/LinearisedQGPV/roger/

Homework 3.6.8: Modify HW 3.6.3.m using (3.6-43). Use (n+1) time step for \(\partial \psi/\partial t = (\psi_{j+1} - \psi_j)/\Delta t\), and re-formulate the matrix; see: LinearisedQGPV-HeatAnalogy*TridiagMatrixEqn3.6-33a.pdf. Repeat (1)-(3) of HW 3.6.2, but still set \(\gamma = 0\) to check that you reproduce HW 3.6.3.

Homework 3.6.9: Repeat HW 3.6.8 with \(\gamma = \pi/4\) (45°); try also \(\gamma = -\pi/4\). Discuss the differences.

Homework 3.6.10: Repeat HW 3.6.4 with \(\gamma = \pi/4\) (45°); try also \(\gamma = -\pi/4\).

Homework 3.6.11: Modify HW 3.6.7.m for non-zero “\(\gamma\)”. Again you need to reformulate the matrix, see: LinearisedQGPV-*CrankNicholson-HW3.6.7.pdf. Repeat HW 3.6.10.

Wind profiles from surface through planetary bouncy layer (PBL) and into the troposphere:
The above solutions (HW3.6.2-3.6.11) give wind high above the sea surface outside the planetary boundary layer (PBL) which is about 1000 m thick, in response to various SST fronts. That wind is represented by \( \psi(x,y) \), the geostrophic stream function (see equation 3.6-43). The geostrophic wind \((u_g, v_g)\) is not a function of height \( z \) from the surface. Within the PBL, wind changes with “\( z \)” given by the Ekman boundary-layer solution which, assuming constant eddy viscosity, is (see (1.C-82)):

\[
\begin{align*}
    u_E &= u_{E0} + \{\beta_c e^{-\xi}/(1+k^4/4)\} \{T_x, \phi_1(\zeta) + T_y, \phi_2(\zeta)\} \quad \text{(3.6-44a)} \\
    v_E &= v_{E0} + \{\beta_c e^{-\xi}/(1+k^4/4)\} \{T_y, \phi_1(\zeta) - T_x, \phi_2(\zeta)\} \quad \text{(3.6-44b)}
\end{align*}
\]

where \( \zeta = z^*/\delta_E \), \( z^* = \) height in \( m \) above the sea surface,
\( \beta_c = \beta_c' /[k.G(k)] = [\alpha'T_o \delta_E g/(k.Uf_o L)] = \epsilon/Fr.k^{-1} = [\psi_o/(k.UL)] = k^{-1}; \)
\( \alpha'T_o \approx 1, \psi_o = g\delta_E/f_o \) (see 3.6-30), \( U = \psi_o/L \) is the velocity scale;
\( u_{E0} = -e^{-\xi}[u_g \cos \zeta + v_g \sin \zeta], \)
\( v_{E0} = e^{-\xi}[u_g \sin \zeta - v_g \cos \zeta], \)
\( \phi_1(\zeta) = [\sin \zeta+(k^2/2).\cos \zeta-e^{(1-k^4)\zeta}]], \)
\( \phi_2(\zeta) = [(k^2/2)\sin \zeta-(\cos \zeta-e^{(1-k^4)\zeta}]], \)
\( (T_x, T_y) = (\partial \psi/\partial x, \partial \psi/\partial y), \) and
\( (u_g, v_g) = (-\partial \psi/\partial y, \partial \psi/\partial x) = k \times \nabla \psi(x,y), \) where \( k \) is the \( z \)-unit vector.

Here, variables are dimensionless (with the exception of those obvious ones on the RHS of the first 2 in (3.6-44c)), and \( \beta_c' \) was previously given after (3.6-26). It is clear that:
\( u_E(x,y,\zeta) \sim 0 \) as \( \zeta \sim \infty \) (i.e. high above the surface, outside the PBL), (3.6-45a) and
\( u_E(x,y,\zeta=0) \sim -u_g(x,y), \) at the surface. (3.6-45b)

Equation (3.6-45b) is the no-slip condition at \( \zeta = 0 \), i.e. the total velocity \( u(x,y,\zeta) = 0 \), where
\( u(x,y,\zeta) = u_g(x,y) + u_E(x,y,\zeta) = k \times \nabla \psi(x,y) + u_E(x,y,\zeta). \) (3.6-46)

Integrating the continuity equation \( \partial w/\partial \zeta = -(\partial \psi/\partial D).(\partial u/\partial x+\partial v/\partial y) = -(\partial \psi/\partial D).\nabla \cdot \mathbf{u}, \) we obtain:
\( w(x,y,\zeta) = w(x,y,0) -(\partial \psi/\partial D).\int_0^{\zeta} \nabla \cdot \mathbf{u} \; d\zeta' = -(\partial \psi/\partial D).\int_0^{\zeta} \nabla \cdot \mathbf{u} \; d\zeta, \) since \( w(x,y,0) = 0. \) (3.6-47)

We derive an analytical formula for “\( w \)” to be used to check the numerically computed vertical profiles. From (3.6-44a), since \( \nabla \cdot \mathbf{u_E} = 0, \) we obtain:
\( \nabla \cdot \mathbf{u_E} = -e^{-\xi}.\sin \xi.\xi E + [\beta_c e^{-\xi}/(1+k^4/4)][\sin \xi+(k^2/2).\cos \xi-e^{(1-k^4)\xi}].\nabla^2 T \) (3.6-48)

Substituting into (3.6-47), the various \( \xi \)-integrals are:
\( \int_0^{\zeta} e^{-\xi} \cos(\zeta') \; d\zeta' = \frac{1}{2} \{1 - e^{-\xi} [\cos(\zeta) - \sin(\zeta)]\} \) (3.6-49)
\( \int_0^{\zeta} e^{-\xi} \sin(\zeta') \; d\zeta' = \frac{1}{2} \{1 - e^{-\xi} [\cos(\zeta) + \sin(\zeta)]\}.

Then, \( w(x,y,\zeta) = -(\delta_E/D)\int_0^{\zeta} \nabla \cdot \mathbf{u_E} \; d\zeta' \)
High above the PBL, $\zeta \sim \infty$, we solve (3.6-12) where

$$w(x,y,\zeta) = (\delta_t/D)(\nabla^2 \psi - \beta_c \cdot \nabla^2 T)/2$$

(3.6-51)\(^n\)

where $\beta_c = \beta_c/(1+k^3/4)$.

\section*{Homework 3.6.12: For each of HW 3.6.8, 9, 10 & 11, calculate $u(x,y,\zeta)$ and $w(x,y,\zeta)$ for all $(x,y)$ points of the model domain, and also for $\zeta_{top} \geq \zeta \geq 0$, where $\zeta_{top} = 5$. Then at $x = -0.5, -1, -2$ and $-4$, in the $y\zeta$-plane for $5 \geq y \geq -5$ & $\zeta_{top} \geq \zeta \geq 0$, plot (a) $u$ as color background and superimpose $(v,w*100)$ vectors. Use 4 panels per page, 1 panel for each of the 4 $x$-sections. Display in each panel the following 6 quantities: “a”, “U” (=u0 in the MATLAB code), “u_max”, “v_max”, “v_min” & “v_min” (note these min/max velocities these are different for different sections). Also plot (b) $(u,v)$ vectors at 4 levels: $\zeta = 0.5, 1, 2 & 4$, also in 4 panels per page. Note that in Roger’s MATLAB program the $w(x,y,\zeta)$ is calculated as: $u = e^{g \nabla \xi} \cdot \nabla \psi$ and $v$ using at least a 2nd-order accurate formula. (c) Finally, to check the $(v,w)$ vectors, compute “w” from the analytical expression (3.6-50)\(^n\), and replot and compare with (a).

\section*{Wind over an ocean eddy:}

Ocean eddies are fronts along their edges. We cannot now assume that $L_x >> L_y$, so cannot assume $\partial^2/\partial x^2 << \partial^2/\partial y^2$. Equation (3.6-33a)\(^n\) needs to be changed to:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + a_1 \cdot \frac{\partial \psi}{\partial x} = S_\psi(x,y),$$

(3.6-52a)\(^n\)

$$S_\psi = G(k)(\partial^2 \psi/\partial x^2 + \partial^2 \psi/\partial y^2)$$

(3.6-52b)\(^n\)

and

$$G(k) = [1-k+k^2/2]/[(k(1+k^3/4)],$$

(3.6-52c)\(^n\)

We solve this by posing it as a time-dependent problem:

$$\frac{\partial \psi}{\partial t} = \nabla^2 \psi + a_1 \cdot \frac{\partial \psi}{\partial x} - S_\psi(x,y)$$

(3.6-53)\(^n\)

where now the “t” is a \textit{real} time not to be confused with the \textit{transformed} variable ‘t’ in (3.6-32a). We solve (3.6-53) by time-stepping until a steady state is reached so that $\partial \psi/\partial t = 0$, so that the final solution “$\psi_{steady}$” is the solution we want for the original time-\textit{independent} equation (3.6-52). An “ADI” (Alternating Direction Implicit) scheme is used:

$$\left(\psi^{n+1/2} - \psi^n\right)/(\Delta t/2) = a_1 \cdot \frac{\partial \psi^{n+1/2}}{\partial x} + \frac{\partial \psi^{n+1/2}}{\partial x} + [\partial \psi^n - S_\psi]$$

(3.6-54a)\(^n\)

$$\left(\psi^{n+1} - \psi^{n+1/2}\right)/(\Delta t/2) = [a_1 \cdot \frac{\partial \psi^{n+1/2}}{\partial x} + \frac{\partial \psi^{n+1/2}}{\partial x}] + \frac{\partial \psi^{n+1}}{\partial y} - S_\psi$$

(3.6-54b)\(^n\)

where

$$\delta_x \psi = (\psi_{i+1,j} - \psi_{i-1,j})/(2\Delta x),$$

$$\delta_{xx} \psi = (\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j})/\Delta x^2$$

and
\[ \delta_{yy}\psi = (\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1})/(\Delta y^2). \]  

(3.6-54c)\textsuperscript{n}

Adding (3.6-54a)\textsuperscript{n} and (3.6-54b)\textsuperscript{n}, we get:

\[ (\psi^{n+1} - \psi^n)/\Delta t = \left[ a^{-1} \delta_x \psi^{n+1/2} + \delta_{xx} \psi^{n+1/2} + [\delta_{yy} \psi^{n+1} + \delta_{yy} \psi^{n}]/2 - S_0 \right] \]

(3.6-55)\textsuperscript{n}

which is seen as a “centered-space” and “centered-time” differencing of (3.6-53)\textsuperscript{n}.

Given \( \psi^{(0)} \) (for \( n=0 \), the term inside the “[...]” on the RHS of (3.6-54a)\textsuperscript{n} is known, and that equation can be solved using the tridiagonal solver. Then after obtaining the \( \psi^{(1/2)} \), the terms inside the “[...]” on the RHS of (3.6-54b)\textsuperscript{n} are known and that equation can be solved for \( \psi^{(1)} \). The process is then repeated (in a “do loop”) to time-step with \( n = 1 \) to solve for \( \psi^{(1+1/2)} \) and \( \psi^{(2)} \), … etc until:

\[ \text{Max}_{i,j}(|\psi^{(N+1)} - \psi^{(N)}|)/\text{Max}_{i,j}(|\psi^{(N+1)}|) < \varepsilon, \text{ for small } \varepsilon = 10^{-3} \text{ say.} \]
3.7 The Eady Problem of Baroclinic Instability (eqn# need to be changed):

Baroclinic instability may occur along special paths “(iii)” across slanting isopycnals (Fig.3.5) which imply the existence of vertical velocity shear by the thermal-wind equations (from geostrophic eqn.(3.6-3a,b) and hydrostatic eqn.(3.6-10b)):

\[
\mathbf{k} \times f_o \partial \mathbf{u} / \partial z = -\nabla b' \tag{3.7-1a}
\]

or

\[
f_o \partial \mathbf{u} / \partial z = \mathbf{k} \times \nabla b' \tag{3.7-1b}
\]

Eady [1949] assumed:

| (i) | \( \beta_o = 0; \) |
| (ii) | \( N^2 \) = constant; |
| (iii) | Basic state has uniform vertical shear: \( U_o(z) = \Lambda z = U.z/H; \) \( \Lambda=\)constant, \( H=\)domain depth & \( U=\Lambda H=\)Basic velocity @\( z=H; \) |
| (iv) | Motion is between 2 rigid horizontal surfaces \( z=0 \) & \( z=H. \) |

(3.7-2a)

Thus the basic-state stream function \( \Psi = -\Lambda yz \) (because \(-\partial \Psi/\partial y = U_o = \Lambda z\)), and the corresponding basic-state \( Q \) (denoted by “\( Q_{basic} \)”) is, from (3.6-16b):

\[
Q_{basic} = 0 \tag{3.7-2b}
\]

Write

\[
\psi(x,y,z,t) = \Psi(y,z) + \phi(x,y,z,t) \tag{3.7-3}
\]

where \( \phi = \) perturbation stream function assumed to be small: \(|\phi|/|\Psi| \ll 1\). The linearized (3.6-16a) is then:

\[
(\partial/\partial t + U_o \partial / \partial x)[\nabla^2 \phi + (f_o^2/N^2).\partial^2 \phi/\partial z^2] = 0 \tag{3.7-4}
\]

Assume:

\[
\phi = \Phi(y,z).e^{ik(x-ct)} \tag{3.7-5}
\]

Then

\[
(\Lambda z - c) \ [\partial^2 \Phi/\partial y^2 + (f_o^2/N^2).\partial^2 \Phi/\partial z^2 - k^2.\Phi] = 0 \tag{3.7-6}
\]

We simplify by assuming 2d vertical plane (xz), so \( \Phi=\Phi(y,z) \), (3.7-6) reduces to:

\[
(\Lambda z - c) \ [(f_o^2/N^2).d^2 \Phi/dz^2 - k^2.\Phi] = 0 \tag{3.7-7a}
\]
Note that the quantity: \( \frac{H^2N^2/f_0^2}{\rho_o/\rho_o} \approx c^2/f_0^2 = R^2 \), can be defined as the Rossby radius of the problem, so that the above equation becomes:

\[
(Az - c) \left( H^2 d^2 \Phi/dz^2 - \mu^2 \Phi \right) = 0, \quad \text{where } \mu^2 = R^2 k^2. \tag{3.7-7b}
\]

Boundary conditions at \( z = 0 \) & \( z = H \) are \( w_1 = 0 \), which using (3.6-11d) is:

\[
d_{/dt}\{\partial \psi/\partial z\} = 0 \tag{3.7-8a}
\]

or, \( cd\Phi/dz + \Lambda \Phi = 0, \quad @z=0 \text{ and } (c - \Lambda H)d\Phi/dz + \Lambda \Phi = 0, \quad @z=H. \tag{3.7-8b}
\]

Equations (3.7-7b) and (3.7-8b) constitute an eigenvalue problem for the eigenvalue \( c \) and eigen function \( \Phi(z) \). The “c” is in general complex, then \( (Az - c) \neq 0 \). The solution to (3.7-7b) is then:

\[
\Phi(z) = A \cosh(\mu z/H) + B \sinh(\mu z/H), \tag{3.7-9a}
\]

so \( d\Phi/dz = A(\mu/H) \sinh(\mu z/H) + B(\mu/H) \cosh(\mu z/H). \tag{3.7-9b} \)

Applying (3.7-8b) (and put \( U = \Lambda H \), see 3.7-2a):

\[
A(U) + B[\mu c] = 0, \tag{3.7-10a}
\]

\[
A[(c-U)\mu \sinh(\mu) + U \cosh(\mu)] + B[(c-U)\mu \cosh(\mu) + Usinh(\mu)] = 0. \tag{3.7-10b}
\]

(3.7-10b) is from 2\(^{nd}\) of (3.7-8b) which gives

\[
(c - \Lambda H) \{A(\mu) \sinh(\mu) + B(\mu) \cosh(\mu)\} + H \Lambda \{A \cosh(\mu) + B \sinh(\mu)\} = 0.
\]

If \( A \) and \( B \) are to be non-zero, then the determinant of (3.7-10a,b) must = 0:

\[
c^2 - U.c + U^2.(\mu \cosh(\mu) - \mu^2) = 0, \tag{3.7-11} \]

which is an equation for “c”. The solution is:

\[
c = (U/2) \pm (U/\mu)\{(\mu/2)^2 - \mu \coth(\mu) + 1\}^{1/2} \tag{3.7-12a}
\]

or

\[
c = (U/2) \pm (U/\mu)\{[\mu/2 - \coth(\mu/2)] [\mu/2 - \tanh(\mu/2)]\}^{1/2} \tag{3.7-12b} \]

using

\[
1 = \coth(\mu/2) \tanh(\mu/2), \quad \text{and } \coth(\mu/2) + \tanh(\mu/2) = \coth(\mu)/2.
\]

\{
\text{since}
\}

\[
\coth(x/2) + \tanh(x/2) = \left[ (e^{x/2} + e^{-x/2})/(e^{x/2} - e^{-x/2}) + (e^{x/2} - e^{-x/2})/(e^{x/2} + e^{-x/2}) \right] = \left[ (e^x + 2 + e^{-x}) + (e^x - 2 + e^{-x}) \right]/(e^x - e^{-x}) = \coth(x/2)
\]
For small “x”, \( \tanh(x) = \frac{(2x + x^3/3! + \ldots)}{(2 + x^2 + \ldots)} = x \cdot \frac{(1 + x^2/3! + \ldots)}{(1 + x^2/2 + \ldots)} < x \). For large x, \( x > \tanh(x) \). Therefore, \( \mu/2 - \tanh(\mu/2) > 0 \) always. For instability, we require that “c” be complex, and the \{..\} in (3.7-12) is negative; therefore:

\[
\mu/2 < \coth(\mu/2), \quad \text{for instability.} \tag{3.7-13}
\]

**Fig.18.7** Graphs of “y=x” and “y=coth(x)” to find intercept \( x \approx 1.2 \) (left panel); and of “y=x” and “y=(coth+tanh)/(coth^2+tanh^2)” to find intercept \( x \approx 0.8 \) (right panel).

The graphs of “x” and “coth(x)” intersect at “x \approx 1.2” (Fig.18.7, left panel). Therefore baroclinic instability occurs if

\[
\mu < \mu_c, \quad \text{where } \mu_c = 2.4 \tag{3.7-14}
\]

Since \( \mu = R\kappa \) (from 3.7-7b),

\[
k < \mu_c/R, \text{ or } \lambda > 2\pi R/\mu_c = 2.6R. \tag{3.7-15}
\]

Growth rate is given by \( \sigma = k\kappa_i \), where \( \kappa_i \) = imaginary part of c; so from (3.7-12), it is:

\[
\sigma = k \frac{(U/\mu)}{[\coth(\mu/2) - \mu/2][\mu/2 - \tanh(\mu/2)]^{1/2}} = \frac{(U/R)}{[\coth(\mu/2) - \mu/2][\mu/2 - \tanh(\mu/2)]^{1/2}} = \sigma_{\text{eady}} \cdot [\coth(\mu/2) - \mu/2][\mu/2 - \tanh(\mu/2)]^{1/2} \tag{3.7-16}
\]

(see Fig.18.8), where \( \sigma_{\text{eady}} \) is often called the Eady growth rate:

\[
\sigma_{\text{eady}} = U/R = U/(NH/f_o) = f_o/Ri^{1/2}. \tag{3.7-17a}
\]
and Ri = NH/U is the Richardson number. (3.7-17b)

**Fig.18.8** Graph of $\sigma/\sigma_{\text{eady}}$ vs. $\mu=kR$, from (3.7-16).

The maximum growth rate is given by setting $d\sigma/d(\mu/2) = 0$, etc, which gives:

$$\mu/2 = \left[\coth(\mu/2) + \tanh(\mu/2)]/\left[\coth^2(\mu/2) + \tanh^2(\mu/2)\right]\right]$$

which is solved graphically (Fig.18.7 right panel), to give:

$$\mu_m = 2 \times 0.8 = 1.6, \quad \text{i.e.} \quad k_m = 1.6/R, \quad \text{i.e.} \quad \lambda_m = 2\pi/k_m = 2\pi R/1.6 = 3.9R.$$  \hspace{1cm} (3.7-18a)

and the maximum growth rate is:

$$\sigma_m = \sigma(\mu=1.6) = 0.31\times\sigma_{\text{eady}}.$$  \hspace{1cm} (3.7-18b)

From (3.7-10a), we get $A/B = -\mu c/U$, so that (3.7-9a) give:

$$\Phi(z)/B = - (\mu c/U).\cosh(\mu z/H) + \sinh(\mu z/H),$$

$$= \sinh(\mu z/H) - (\mu c/U).\cosh(\mu z/H) - i.(\mu c/U).\cosh(\mu z/H)$$  \hspace{1cm} (3.7-19)

where from (3.7-12):

$$c_r/U = (1/\mu)\left\{\left[ \coth(\mu/2) - \mu/2 \right] \cdot \left[ \mu/2 - \tanh(\mu/2) \right] \right\}^{1/2}$$

and $$c_i/U = \frac{1}{2}.$$
The structure of the Eady mode can be seen in Fig.18.10. The perturbation stream function (top panel) shows that bottom perturbations lead those at top, so that perturbed flow leans against the mean current; intuitively, the perturbed flow tends to retard the mean. This kind of structure is generally true for baroclinic (and barotropic, as we see below) instability, not just for the Eady problem.
4. Frontogenesis by wind waves?

Equations of an incompressible fluid assuming hydrostacy (see 2.2-1):

\[
\nabla \cdot \mathbf{u} = 0 \tag{4.1-1a}
\]
\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{f}_a \times \mathbf{u} = -\nabla \Phi + \frac{\partial (K_M \partial \mathbf{u}/\partial z)}{\partial z} \tag{4.1-1b}
\]

where \( \nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \), \( \mathbf{u} = (u,v,w) \), \( \mathbf{f}_a = 2\Omega + \mathbf{\xi} \) is the absolute vorticity, \( \Omega = \) earth’s rotation vector, \( \mathbf{\xi} = \nabla \times \mathbf{u} \) is the relative vorticity vector, \( g \) is acceleration due to gravity, \( p \) is pressure, \( \rho \) is density, \( K_M \) is the kinematic eddy viscosity (unit: \( m^2 s^{-1} \)), and \( \Phi = gz + p/\rho + |\mathbf{u}|^2/2 \).

Write \( \mathbf{u} = \mathbf{U} + \mathbf{u}_s \), where \( \mathbf{u}_s = \) Stokes velocity, then assuming \(|\partial \mathbf{U}/\partial t| >> |\partial \mathbf{u}_s/\partial t|\), \(|\partial \mathbf{U}/\partial z| >> |\partial \mathbf{u}_s/\partial z|\) and \(|\mathbf{U}| >> |\mathbf{u}_s|\) (justified perhaps by phase-averaging?), we have:

\[
\frac{\partial \mathbf{U}}{\partial t} + \mathbf{f}_a \times \mathbf{U} = \mathbf{u}_s \times \mathbf{\xi} - \nabla \Phi + \frac{\partial (K_M \partial \mathbf{U}/\partial z)}{\partial z} .
\]

Since \( \mathbf{u}_s \sim e^{kz} \), the force \( \mathbf{u}_s \times \mathbf{\xi} \) is concentrated near the surface. But it appears only as a body force. Please think and work out the Ekman solution that includes this term (ref: Polton et al 2005). Then see if the following frontogenesis might be feasible:

\[\text{SST}\]

\[\text{Wind: } \mathbf{u}_s\]

\[\mathbf{u}_s \times \mathbf{\xi}\]

Front sharpens due to flow convergence by \( \mathbf{u}_s \times \mathbf{\xi} \)?
Fig. 18.10 Structure of most unstable Eady mode.
Appendix to Chapter 2: some generalizations. PV-conservation and integral constrains

We first derive the conservation of *absolute* vorticity (= background + relative) in general terms; then we specialize to *reduced-gravity* equation and apply it to basins.

**A2.1. General conservation of absolute vorticity**

The continuity and momentum (fixed to the rotating earth) equations are:

\[ \nabla \cdot \mathbf{u} = 0 \]  \hspace{1cm} (A2.1-1a)

\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{f}_a \times \mathbf{u} = -\nabla (gz + p/\rho_o + |\mathbf{u}|^2/2) - r\mathbf{u} \]  \hspace{1cm} (A2.1-1b)

where \( \nabla = (\partial / \partial x, \partial / \partial y, \partial / \partial z) \), \( \mathbf{u} = (u,v,w) \), \( \mathbf{f}_a = 2\mathbf{\Omega} + \mathbf{\xi} \) is the absolute vorticity, \( \mathbf{\Omega} \) = earth’s rotation vector, \( \mathbf{\xi} = \nabla \times \mathbf{u} \) is the relative vorticity vector, \( r \) is the (constant linear) friction coefficient, and we have used

\[ \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{\xi} \times \mathbf{u} + \nabla (|\mathbf{u}|^2/2) \]  \hspace{1cm} (A2.1-2)

to rewrite the material derivative \( D/Dt \) in (A2.1-1b). Also, let \( \mathbf{x} \) be defined such that

\[ \mathbf{f}_a = (\xi_x, \xi_y, f+\zeta) \]  \hspace{1cm} (A2.1-3)

where \( \zeta = k.\nabla \times \mathbf{u} \) is the z-component of the relative vorticity. Note for large-scale flows: \( \mathbf{f}_a \approx (0, 0, f+\zeta) = k(f+\zeta) \), but this is not assumed here.

Take the curl \( \nabla \times (A2.1-1b) \):

\[ \frac{\partial \mathbf{\xi}}{\partial t} + \nabla \times (\mathbf{f}_a \times \mathbf{u}) = -r\mathbf{\xi} \]  \hspace{1cm} (A2.1-4)

To evaluate the second term, we use the following formula that for any 2 vectors \( \mathbf{A} \) and \( \mathbf{B} \):

\[ \nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \nabla \cdot \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{A} + \mathbf{B} \nabla \mathbf{A} - \mathbf{A} \nabla \mathbf{B} \]  \hspace{1cm} (A2.1-5)

Thus,

\[ \nabla \times (\mathbf{f}_a \times \mathbf{u}) = \mathbf{f}_a \nabla \cdot \mathbf{u} - \mathbf{u} \nabla \cdot \mathbf{f}_a + \mathbf{u} \nabla \mathbf{f}_a - \mathbf{f}_a \nabla \mathbf{u} \]

\[ = 0 - \mathbf{u} \nabla \cdot \mathbf{\xi} + \mathbf{u} \nabla \mathbf{f}_a - \mathbf{f}_a \nabla \mathbf{u} \]  \hspace{1cm} (1st uses (A2.1-1a), and 2nd term is because \( \nabla \cdot \mathbf{f} = 0 \))

\[ = \mathbf{u} \nabla \mathbf{f}_a - \mathbf{f}_a \nabla \mathbf{u} \]  \hspace{1cm} (since \( \nabla \cdot \mathbf{\xi} = \nabla \cdot (\nabla \times \mathbf{u}) \equiv 0 \))  \hspace{1cm} (A2.1-6)

Then (A2.1-4) becomes:

\[ \frac{\partial \mathbf{f}_a}{\partial t} + \mathbf{u} \nabla \mathbf{f}_a = \mathbf{f}_a \nabla \mathbf{u} - r\mathbf{\xi} \]  \hspace{1cm} (A2.1-7)
The k-component is:

$$\partial (f+\zeta)/\partial t + \mathbf{u} \cdot \nabla (f+\zeta) \equiv D(f+\zeta)/Dt = (f+\zeta)\partial w/\partial z - r\zeta$$ (A2.1-8)

Consider the meaning of $(f+\zeta)\partial w/\partial z$. When there is stretching (squashing) $\partial w/\partial z > 0 (< 0)$, following a fluid parcel, the magnitude of z-component absolute vorticity increases, assuming $|\zeta| < |f|$ so that sign$(f+\zeta) = \text{sign}(f)$. This statement is true irrespective of the sign of “f.” In slow $(\partial/\partial t \approx 0)$, large-scale $(|\zeta| \ll |f|)$ flows in the ocean or atmosphere, stretching (squashing) produces poleward (equatorward) flows, i.e. $v \sim \text{sign}(f)$. Show this - homeworkA2.1, which is Sverdrup relation. If $f \approx \text{constant}$, then fluid parcel acquires cyclonic (anticyclonic) vorticity $\delta\zeta \sim \text{sign}(f)$ ($\delta\zeta \sim \text{sign}(f)$) when stretched (squashed).

The friction term “$-r\zeta$” always tends to slow down spinning.

We now rewrite (A2.1-8) using the horizontal velocity and gradient vectors: $\mathbf{u}_h = (u, v)$ and $\nabla_h = (\partial/\partial x, \partial/\partial y)$. Using $\nabla \cdot \mathbf{u} = 0$, the 2nd term:

$$\mathbf{u} \cdot \nabla (f+\zeta) = \nabla_h \cdot [\mathbf{u}_h (f+\zeta)] = \nabla_h \cdot [\mathbf{u}_h (f+\zeta)] + \partial [w(f+\zeta)]/\partial z$$,

Then (A2.1-8) becomes:

$$\partial (f+\zeta)/\partial t + \nabla_h \cdot [\mathbf{u}_h (f+\zeta)] = -\partial [w(f+\zeta)]/\partial z + (f+\zeta)\partial w/\partial z - r\zeta.$$ (A2.1-9)

In this form, we see the appearance of the first 2 terms on the RHS which represent different physical processes. To understand, integrate over $z$ from $z_B$ to $z_T$:

$$\int_{z_B}^{z_T} \left[ \nabla_h \cdot [\mathbf{u}_h (f+\zeta)] \right] dz$$

$$= -\{[w(f+\zeta)]_{z_T} - [w(f+\zeta)]_{z_B}\} + \int_{z_B}^{z_T} (f+\zeta) \partial w/\partial z dz - \int_{z_B}^{z_T} r\zeta dz$$ (A2.1-10)

Therefore the $z$-integral of “$-\partial [w(f+\zeta)]/\partial z$” is term $Q_{fa}$ which expresses the vertical transport of absolute vorticity into (if $Q_{fa} > 0$) or out (if $Q_{fa} < 0$) of the layer of thickness $z_T - z_B$. Suppose there is downwelling across the top boundary $z=z_T$ and upwelling or weaker downwelling across the bottom $z=z_B$, then the absolute vorticity of the layer would tend to increase with time. On the other hand, $S_{fa}$ is the stretching term as before. If we linearize assuming that $|\zeta| < |f|$, then the first 2 terms on the RHS of (A2.1-9) cancel; the vertical velocity then does not contribute to the budget of vorticity, say if one further integrates (A2.1-10) over area to form volume integral of a closed basin.

**A2.2. Conservation of mass equation of a layer bounded by 2 isopycnals with mixing**
Here we generalize mass conservation $\partial \eta / \partial t + \nabla . U = 0$ to a layer bounded by 2 isopycnals across which there may be with mixing. Consider a layer of fluid bounded by 2 isopycnal surfaces (i.e. $\rho = \text{constant}$) above ($z = z_T(x,t)$) and below ($z = z_B(x,t)$); see the middle layer#2 in Fig.A2.2-1.

Integrate $\nabla . \mathbf{u} = 0$, where $\mathbf{u}_{3d} = (u, w)$ and $\mathbf{u} = (u, v)$, across the layer and integrate by parts:

$$\int_{z_B}^{z_T} \nabla . \mathbf{u}_{3d} \, dz \equiv \nabla . \mathbf{U} - \mathbf{u}_T . \nabla z_T + \mathbf{u}_B . \nabla z_B + w_T - w_B = 0$$  \hspace{1cm} (A2.2-1a)

where $\mathbf{U} = \int_{z_B}^{z_T} (u, v) \, dz = (u, v) \, h; \quad h = z_T - z_B$.  \hspace{1cm} (A2.2-1b)

The second form $\mathbf{U} = (u, v) h$ is because the velocity within each layer is $z$-independent. Alternatively we can think of a depth-averaged velocity $\mathbf{\bar{u}}$ within the layer. To evaluate $w_T$ and $w_B$, consider:

$$D\rho /Dt = \partial (K_h \partial \rho / \partial z) / \partial z,$$  \hspace{1cm} (A2.2-2a)

where $K_h$ is eddy diffusivity [see Gill 1982]. This equation is approximated as:

$$\partial \rho / \partial t + \mathbf{u} . \nabla \rho + w \rho_0 / dz = \partial (K_h \partial \rho / \partial z) / \partial z.$$  \hspace{1cm} (A2.2-2b)

where $\rho_0(z)$ is the ambient density profile. Note that here, the gradient operator $\nabla = (\partial / \partial x, \partial / \partial y)$ and the velocity $\mathbf{u} = (u,v)$ are two-dimensional. Divide (A2.2-2b) by $\rho_0 = d\rho_0/dz$ and approximating $\delta \rho \approx -\delta Z_\rho (d\rho_0/dz)$, where $Z_\rho$ = isopycnal depth measured from surface (i.e. at $z = -Z_\rho$):
\[-\partial Z_p/\partial t - \mathbf{u} \cdot \nabla Z_p + w = (\rho_{oc})^{-1} \frac{\partial}{\partial z} (K_h \rho \partial \zeta / \partial z). \quad \text{(A2.2-3)}\]

Then
\[
\begin{align*}
w_T &= \partial z_T/\partial t + \mathbf{u}_T \cdot \nabla z_T + (\rho_{oc})^{-1} \left[ \partial (K_h \rho \partial \zeta / \partial z) \right]_T, \\
w_B &= \partial z_B/\partial t + \mathbf{u}_B \cdot \nabla z_B + (\rho_{oc})^{-1} \left[ \partial (K_h \rho \partial \zeta / \partial z) \right]_B.
\end{align*}
\quad \text{and} \quad \text{(A2.2-4a)}
\]

Substitute into (A2.2-1a):
\[
\nabla (\mathbf{u} h) + \partial h/\partial t = -(\rho_{oc})^{-1} \{ \left[ \partial (K_h \rho \partial \zeta / \partial z) \right]_T - \left[ \partial (K_h \rho \partial \zeta / \partial z) \right]_B \}
\quad \text{(A2.2-5)}
\]

We approximate the RHS \{..\} by finite-differences, and Taylor-expand (Fig.A2.2.1):
\[
\begin{align*}
[\partial (K_h \rho \partial \zeta / \partial z) / \partial z]_T &\approx [K_{h1} \Delta \rho_1/(H_1 \eta_1) - K_{h2} \Delta \rho_2/(H_2 + \eta_1 \eta_2)] / (H_1 + H_2)/2 \\
&\approx [2/(H_1 + H_2)] \{ (K_{h1} \Delta \rho_1/H_1) - (K_{h2} \Delta \rho_2/H_2) - [K_{h1} \Delta \rho_1/H_1^2 + K_{h2} \Delta \rho_2/H_2^2] \eta_1 + K_{h2} \Delta \rho_2/H_2^2 \eta_2 \} + .. \quad \text{(A2.2-6)}
\end{align*}
\]

We assign an entrainment velocity \(w_e\) to the RHS of (A2.2-5), so that using (A2.2-6):
\[
\begin{align*}
w_e &\approx [2/(H_1 + H_2)] \{ (K_{h1} \Delta \rho_1/H_1) - (K_{h2} \Delta \rho_2/H_2) - [K_{h1} \Delta \rho_1/H_1^2 + K_{h2} \Delta \rho_2/H_2^2] \eta_1 + K_{h2} \Delta \rho_2/H_2^2 \eta_2 \}/\rho_{oc}
\end{align*}
\]

But \(w_e\) must = 0 when both \(\eta_1\) and \(\eta_2\) = 0, which requires that \((K_{h1} \Delta \rho_1/H_1) - (K_{h2} \Delta \rho_2/H_2) = 0\), and:
\[
\begin{align*}
w_e &\approx [-2/(H_1 + H_2)] \{ [K_{h1} \Delta \rho_1/H_1^2 + K_{h1} \Delta \rho_1/(H_1 H_2)] \eta_1 - K_{h1} \Delta \rho_1/(H_1 H_2) \eta_2 \}/\rho_{oc} \\
&= [-2/(H_1 + H_2)](K_{h1} \Delta \rho_1/H_1^2)((1 + (H_1/H_2)(\eta_1 - \eta_2)))/\rho_{oc} = -2(\rho_{oc}^{-1})K_{h1}(\eta_1 - \eta_2) \Delta \rho_1/(H_2 H_1^2)
\end{align*}
\]

Therefore,
\[
\begin{align*}
w_e &= \kappa(\eta_1 - \eta_2) = \kappa \eta
\end{align*}
\quad \text{(A2.2-7a)}
\]

where \(\kappa = -2(\rho_{oc}^{-1})K_{h1} \Delta \rho_1/(H_2 H_1^2)\), and \(\eta_1 - \eta_2\)

is just the change in the thickness of \(z_T - z_B = h\).

Finally, (A2.2-5) gives:
\[
\begin{align*}
\partial h/\partial t + \nabla (\mathbf{u} h) &= -\kappa \eta + Q \quad \text{(A2.2-8)}
\end{align*}
\]

where \(Q\) = mass flux which may be additionally specified. Physically, \(w_e\) produces flux through water-mass conversion by turbulence. However, from the above derivation (see A2.2-5) it is conceivable that other processes can additionally produce \(Q\) (e.g. by air-sea interaction which produces cold-water mass). Equation (A.2.3-8) with \(h = D\) and \(\kappa = 0\) is the same as (2.3-5).
A2.3. Reduced-gravity model

In Chapter E (see equation (1.E-31)), we derived the reduced-gravity equation which is:

\[
\begin{align*}
Dh/ Dt + h \nabla \cdot \mathbf{u} &= Q - \kappa \eta \\
\partial \mathbf{u}/ \partial t + \mathbf{k}(f+\zeta) \times \mathbf{u} &= - \nabla (g' h + |\mathbf{u}|^2/2) + \nabla H - \mathbf{r}u
\end{align*}
\]  

(A2.3-1a) \hspace{1cm} (A2.3-1b)

where \( h = H + \eta, \) \( H(x,y) = \) mean water depth, \( \eta(x,y,t) = \) layer-anomaly (i.e. deviation from \( H), \) \( g' = g\Delta \rho/\rho_0 \) is the reduced gravity, \( \tau = \) kinematic interfacial stress, and \( \kappa \) is the (constant) Newtonian cooling coefficient. In this section, the gradient operator \( \nabla = \partial/ \partial x, \partial/ \partial y \) and the velocity \( \mathbf{u} = (u,v) \) are two-dimensional. Equation (19.3-1a) is the same as (19.2-8). As explained before, the “\( Q \)” represents a prescribed mass source, usually concentrated in a small region. For the deep ocean, \( Q \) may represent deep-water formation due to cooling [in the N-Atlantic, say the Labrador Sea; Stommel 1958]. In the lower atmosphere, \(-Q\) may represent heating as over the equatorial region [Gill, 1980]. We can think of \( h \) as the thickness of a lower layer on top of which there is an infinitely thick upper layer; so in Fig.1.E-5, we consider layer2 only so that \( H_2 = H \) and \( \eta_2 = \eta; \) the \( \tau \) would then be \( \approx 0. \) The Newtonian cooling \(-\kappa \eta\) may be given a physical interpretation in terms of mixing of density anomaly which tends to erase that anomaly. Suppose the interface locally develops an upward dome (like an upside-down salad bowl), \( \eta > 0, \) so that there is an excess of lower-layer water at the interface. Mixing would tend to erase that anomaly and, in the model, is accomplished by upward-transport (cross-interfacial velocity \( w_i > 0 \) of lower-layer fluid into the upper layer, causing the interface to drop, i.e. \( \partial \eta/ \partial t \sim -\kappa \eta. \) Thus \( w_i = \kappa \eta \) [e.g. Kawase 1987, JPO].

As before, take the curl \( \nabla \times (A2.3-1b): \)

\[
\mathbf{k} \ \partial \zeta/ \partial t + \nabla \times [\mathbf{k}(f+\zeta) \times \mathbf{u}] = (\nabla \times \tau H - r\zeta) \mathbf{k}
\]

(A2.3-2)

As before, use the vector identity (A2.1-5) to evaluate the 2\textsuperscript{nd} term: put \( \mathbf{A} = \mathbf{k}(f+\zeta) \) and \( \mathbf{B} = \mathbf{u}: \)

\[
\begin{align*}
\nabla \times [\mathbf{k}(f+\zeta) \times \mathbf{u}] &= \mathbf{k}(f+\zeta) \nabla \cdot \mathbf{u} - \mathbf{u} \nabla \cdot \mathbf{k}(f+\zeta) + \mathbf{u} \nabla \mathbf{k}(f+\zeta) - \mathbf{k}(f+\zeta) \nabla \mathbf{u} \\
&= \mathbf{k} \{(f+\zeta) \nabla \cdot \mathbf{u} \} - 0 + \mathbf{k} \{ \mathbf{u} \nabla (f+\zeta) \} - 0 \\
&= \mathbf{k} \{ (f+\zeta) [(-\partial h/ \partial t - \mathbf{u} \cdot \nabla h) + Q - \kappa \eta]/h + \mathbf{u} \nabla (f+\zeta) \}
\end{align*}
\]

(A2.3-3)

Substitute into \( h^2 \times (A2.3-2), \) and write \( \partial \zeta/ \partial t \equiv \partial(f+\zeta)/ \partial t, \) we obtain (Show this - homeworkA2.3):

\[
D[(f+\zeta)/h]/Dt = [(f+\zeta)/h]^2[(-Q+\kappa \eta)] + (\nabla \times \tau H - r\zeta)/h
\]

(A2.3-4a)

or  \( h.D(PV)/Dt = PV.(Q-\kappa \eta) + \nabla \times \tau H - r\zeta \)

(A2.3-4b)

where  \( PV = (f+\zeta)/h \)

(A2.3-5)

is the potential vorticity of the 1-layer reduced-gravity model (A2.3-1). In the absence of source \( Q \) and friction, (A2.3-4) says that \( PV \) of a fluid parcel is conserved.
Using (A2.3-1a), equation (A2.3-4b) can be rewritten as:

\[
\frac{D(h.PV)}{Dt} = - (h.PV) \nabla \cdot \mathbf{u} + \nabla \times \mathbf{\tau} H - r \zeta
\]

or

\[
\partial (h.PV)/\partial t + \nabla . (u h.PV) = \nabla \times \mathbf{\tau} H - r \zeta
\]

i.e.

\[
\partial (f+\zeta)/\partial t + \nabla . [u (f+\zeta)] = \nabla \times \mathbf{\tau} H - r \zeta
\]

(A2.3-6a)

(A2.3-6b)

Integral constraints in closed or semi-closed basins:

We now consider a basin, which is either completely closed or can have some openings or passages; the situation applies to e.g. a deep ocean basin (in which case \( \tau \approx 0 \)). We also consider steady state or slowly time-varying \((f+\zeta), \partial/\partial t \approx 0 \) in (A2.3-6b):

\[
\nabla . [u (f+\zeta)] = \nabla \times \mathbf{\tau} H - r \zeta
\]

(A2.3-7a)

\[
\nabla . [U PV] = \nabla \times \mathbf{\tau} H - r \zeta, \text{ where } U = u h.
\]

(A2.3-7b)

Then integrate over the entire basin \( S \):

\[
\oint S \nabla . [U PV] dxdy = \oint S \nabla \times \mathbf{\tau} H dxdy - \oint S r \nabla \times \mathbf{u} dxdy
\]

\[
\oint_C (U. n) PV ds = \oint_C \left( \frac{\mathbf{\tau}}{H} - r \mathbf{u} \right) . l ds \quad \text{(apply Gauss & Stokes theorems)}
\]

(A2.3-8)

where \( C \) is the bounding circuit (boundary) around the basin \( S \), \( n \) and \( l \) are unit vectors normal and tangential, respectively, to the boundary; \( n \) is positive outward across \( C \) and \( l \) is positive anticlockwise along \( C \). Let \( C \) have \( N \) openings (i.e. straits), then the LHS = 0 except at those openings, so that:

\[
\sum_{i=1}^{N} \int_{C_{1i}}^{C_{2i}} (U . n) \left( \frac{\mathbf{\tau}}{H} - r \mathbf{u} \right) . l ds = \oint_C \left( \frac{\mathbf{\tau}}{H} - r \mathbf{u} \right) . l ds
\]

(A2.3-9)

We can get rid of the “\( \zeta \)” term from the LHS by noting that along \( C \), the vorticity is (make a sketch then you will see):

\[
\zeta = -d(u.n)/ds \quad \text{along } C.
\]

(A2.3-10)

Then

\[
\int_{C_{1i}}^{C_{2i}} (U . n) \left( \frac{\mathbf{\tau}}{H} \right) ds = \int_{C_{1i}}^{C_{2i}} (u.n) \zeta ds = - \int_{C_{1i}}^{C_{2i}} \frac{d(u.n)^2}{2} = [(u.n)^2|_{C_{1i}} - (u.n)^2|_{C_{2i}}]/2
\]

(A2.3-11)
So, if $u.n$ is small along the two side walls of the strait, say $= 0$ for no-slip condition, or if the strait is narrow enough that the $u.n$ is approximately uniform, then the contribution of this $\zeta$-flux term is small. Then letting also $h \approx H$, and $f$ varies little over the narrow strait, (A2.3-9) becomes:

$$\sum_{i=1}^{N} q_i f_i / H_i \approx \frac{\phi_c}{H} \left( \frac{\zeta}{H} - ru \right) \cdot l \, ds$$  \hspace{1cm} (A2.3-12)

where

$$q_i = \int_{c_{1i}}^{c_{2i}} (U.n) ds$$ \hspace{1cm} (A2.3-13)

is the volume flux through the strait “i,” positive outflux from the basin. Rearranging, (A2.3-12) becomes:

$$\phi_c ru \cdot l \, ds \approx \frac{\phi_c}{H} \left( \frac{\zeta}{H} \right) \cdot l \, ds - \sum_{i=1}^{N} q_i f_i / H_i$$ \hspace{1cm} (A2.3-14)

Thus the basin’s circulation is controlled by (i) distribution of $\tau$ (windstress if the layer is the upper ocean), and (ii) influx and/or outflux of the ambient $PV = f/H$ through the various openings.

**South China Sea (SCS):**

We use SCS as an example illustrating the above ideas. The sill depth of Luzon Strait is 2000 m while the sill depth of the Mindoro/Panay Strait is shallower at 570 m. Elsewhere in SCS, the water depth is shallower than 120 m (Fig.A2.3-1).

Fig.A2.3-1 Topography of SCS shaded for water depths shallower than, left: 120m showing 2 openings (Luzon and Mindoro-Panay (near 12N & 121E) Straits), middle: shallower than 570 m showing 1 opening (Luzon Strait), and right: shallower than 2000 m showing that SCS is completely closed below the sill depth of the Luzon Strait at 2000m. (From Xu & Oey 2014).

We divide SCS into 4 layers. The surface layer (layer 0) is $0 > z > -120$ m which is considered to be driven by eddies and other fast and energetic processes. We do not consider this layer.
Layer#1:

At the base of the surface layer is Ekman pumping by $\nabla \times \tau$ which acts on the layer 1 below (the upper layer which we now consider) which is $-120 \text{ m} > z \geq -570 \text{ m}$; therefore $H_1 = 450 \text{ m}$. Layer 1 has 2 openings, one at Luzon with $q_L f_L / H_1 < 0$ (influx) and the other one at Mindoro Strait with $q_M f_M / H_1 > 0$ (outflux; see Fig.A2.3-2). Here subscripts “$L$” and “$M$” denote Luzon and Mindoro respectively, and “1” denotes layer 1. From the numerical model, we find that $|q_M| < |q_L|$, and since $f_M < f_L$, influx of the layer 1 PV into SCS exceeds outflux. The net contribution of PV fluxes, $-\sum_{i=1}^{N} q_i f_i / H_i$, to the basin’s circulation $\oint_C \mathbf{r} \cdot \mathbf{u} \cdot \mathbf{l} \, ds$ is therefore cyclonic. In SCS, the mean wind stress has a positive curl and $\oint_C \left( \frac{\tau}{H} \right) \cdot \mathbf{l} \, ds$ is also cyclonic. The resulting basin’s circulation in the upper layer is therefore cyclonic, as sketched in the left panel of Fig.A2.3-2.

![Layer#1](image)

Fig.A2.3-2 Left to right: schematic sketches of SCS circulation in upper, middle and lower layers. (From Xu & Oey 2014).

Layer#2:

Here the basin only has 1 opening in the Luzon Strait where numerical model shows an outflow and therefore the basin circulation is anticyclonic (Fig.A2.3-2 middle panel).

Layer#3:

Here the basin is closed. Numerical model shows deep eddying gyres of alternating signs, since $\oint_C \mathbf{r} \cdot \mathbf{u} \cdot \mathbf{l} \, ds$ must = 0. The large-scale circulation is shown in Fig.A2.3-2 (right panel).
Appendix 3: How to run
the mpi-version of the Princeton Ocean Model – mpiPOM

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0. Prologue

The directory [ftp://profs.princeton.edu/leo/mpipom/atop/tests/] contains 2 idealized test cases:
exp001 -- topographic (barotropic) standing wave in an f-plane channel
exp002 -- dambreak (warm_South & cold_North) in f-plane zonal channel

To start learning how to use, check the runscript batch*.csh
Also see Section 1 below for descriptions of model parameters.

These test cases are self-contained requiring NO input data to run
They have identical codes: pom.h & pom/*.f & pom/*.F
They are set up to use 4 processors only

The codes are general, however, and are also to be used for realistic simulations in which case you will need the grid generation codes in [ftp://profs.princeton.edu/leo/mpipom/atop/eastpac/gridf/]
in order to generate all the necessary input files That grid-generation is for a small region in the Eastern Pacific – see input file examples in that directory.

I have used the code for realistic (in my case North Pacific) simulations.

The code contains features such as:

(1) particle-tracking (we call them "drifters");
(2) passive tracer simulation (to simulate e.g. oil spill);
(3) Stokes drift corrections;
(4) wave-enhanced mixing;
(5) satellite SSHA assimilation;
(6) drifter-assimilation; and other things.

See:

You may also want to access a class I am teaching using mpiPOM:
http://oeylectures.pbworks.com/w/page/69115711/FrontPage
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1. The Runscript -- batch*.csh
We are designing mpiPOM so that this “control file” is one which you would likely need to change to run new cases.

Start-Up Parameters:

date_startfix reference date (default 1987-07-02) all dates are referenced to this
date_start0 start date (default 2012-01-17)
days simulation days from date_start0
date_end0 end date of simulation
title run title
windf type of wind, used only for realistic case (default "gfsw")
num_of_nodes (default = 4)
run_time (default = "99:00:00")

Control Flags:

ngrid <0 for test cases to specify grid in initialize.f; =1 for realistic runs
tsfreeze =1 to read/use T/S climatology (e.g. WOA), otherwise =0
wind =1 to read/use realistic wind data, otherwise =0
river =1 to read/use realistic river data, otherwise =0
assimssh =1 to turn on satellite SSHA assimilation, otherwise =0
assimsst =1 to turn on satellite SST assimilation, otherwise =0
assimdrf =1 to turn on drifter assimilation, otherwise =0
tide =1 to turn on tides, otherwise =0
trajdrf =0 no particles to track; =1 (or 2) position@t=0 specified (or read drf.list)
stokes =1 for Stokes currents and vortex force, otherwise =0
vort =1 to do vorticity analysis, otherwise =0
rst_flag =1 to read from a “restart” file from a previous run, otherwise =0

Model Domain Dimensions (dim.h -- using Fig.1 example):

kb # vertical sigma levels (default = 3)
im_global # x-grid cells for modeled domain (=14 in Fig.1)
jm_global # y-grid cells for modeled domain (=17 in Fig.1)
im_local # x-grid cells for each processor (=6 in Fig.1)
jm_local # y-grid cells for each processor (=7 in Fig.1)
n_proc # processors (i.e. num_of_nodes) (=9 in Fig.1)

Note:
1. n_proc=[(im_global-2)/(im_local-2)]*[jm_global-2]/(jm_local-2)]; see pom.h
2. In the batch*.csh, im_global=202, jm_global=47, im_local=52, jm_local=47 and n_proc=4 for text cases exp001 (kb=3) and exp002 (kb=51)
**Fig. 1** Example of data decomposition scheme for a global size of $14 \times 17$, local sizes of $6 \times 7$, and $9$ processors ($n_{\text{proc}}=9$). Crosses represent the boundaries of the global domain, shadow areas are the ghost cells (or local boundaries) and arrows indicate the communication between local domains to exchange variables at the ghost cells. (From Jordi A., D.-P. Wang, 2012: sbPOM: A parallel implementation of Princeton Ocean Model, Environmental Modeling & Software, 38, December 2012, Pages 59-61, ISSN 1364-8152, 10.1016/j.envsoft.2012.05.013.)

**Model Namelist (pom.nml):**

**mode**

=3 (default) or 2 for 3D or 2D; =4 for diagnostic modeling with T&S fixed

**nadv**

=1 (default) for center-space-difference advection, =2 for positive-definite

**nitera**

=3 (default) used only if nadv=2

**sw**

=0.98 (default) used only if nadv=2

**npg**

=1 (default) pressure gradient scheme 1->2nd order; 2->4th order

**dte**

=10. (seconds) external time step to solve for 2D, surface elevation etc.

**isplit**

=45; isplit*dte = dti internal time step to solve 3D fields

**relxs**

=1.157407407e-5 (1/sec); surface relaxation time constant for T&S

**relx**

=1.586e-8 (1/s); 3D relaxation time constant for T&S

** relixd**

=5.e-4 (1/m); 3D relaxation inverse-depth scale for T&S

**prtd1**

=day interval & also averaging period of 3D (‘title’*.nc) outputs

**prtd2**

=day interval & also averaging period of 2D (SRF*.nc & vor*.nc) outputs

**ipex**

=1 for $x$-periodic (default is 0)

**ipery**

=1 for $y$-periodic (default is 0)

**nld**

=1 for 1D $z$-simulation (default is 0)

**np**

=# particles released = trajdrf – in “Control Flags” above

**Model (Logical) Switches (switch.nml):**

**calc_**

These are already defined by “Control Flags” above

There are fixed parameters for assimdrf=1 & also trajdrf>0 (see batch*.csh) – no need to change.
Appendix 4: Surface flux conditions

I often received questions about the signs of surface fluxes (momentum, heat and mass) in POM or mpiPOM – why they are apparently reversed. Actually they are not; Prof. Mellor had good reasons to call the variables e.g. $w_{usurf} = -$ wind stress (divided by $\rho_o$) in x-direction, etc.

The Reynolds averaged equations [Hinze 1961] are derived by splitting the total variable (denoted by an upper case) “$R$” into a mean (denoted by lower case) “$r$” and a turbulence (primes) $r'$:

$$R = r + r'; \quad <R> = r; \quad <r'> = 0$$

(A4.1)

where $<..> = \text{mean}$. The x-momentum equation is:

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + W \frac{\partial U}{\partial z} = \ldots$$

Using $\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0$, the above becomes:

$$\frac{\partial U}{\partial t} + \frac{\partial (UU)}{\partial x} + \frac{\partial (UV)}{\partial y} + \frac{\partial (UW)}{\partial z} = \ldots$$

Now use (A4.1) and then take the mean to obtain the Reynolds averaged equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \ldots + \partial <u'w'>/\partial z + \{\partial <u'u'>/\partial x + \partial <u'v'>/\partial y\}$$

$$\Rightarrow \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \ldots + \partial \tau^x/\partial z + \text{HorMix}$$

where HorMix is the horizontal mixing term modeled in POM by the “AAM*..” which involves “horcon”; and

$$\tau^x = <u'w'>$$

(A4.2)

is the Reynolds stress in the x-direction. In POM (and nearly all other models), this is modeled as:

$$\tau^x = <u'w'> = K_M \partial u/\partial z.$$ 

(A4.3)

At the sea-surface, $\tau^x$ is simply the wind stress in the x-direction:

$$\tau^{x_0} = <u'w'>|_{z=\eta} = -w_{usurf}(i,j).$$

(A4.4)

Similarly, $\tau^{y_0} = -w_{vsurf}$.

For heat flux [see section 4b of Oey 1986]:

$$K_M \partial T/\partial z|_{z=\eta} = -w_{tsurf}(i,j),$$

(A4.5)

so that since heating in general leads to POSITIVE $\partial T/\partial z$, then it also corresponds to NEGATIVE $w_{tsurf}$.

Similarly, for mass flux:

$$K_M \partial S/\partial z|_{z=\eta} = -w_{ssurf}(i,j) = -(P-E),$$

(A4.6)

since evaporation ($E > 0$ always) leads to POSITIVE $\partial S/\partial z$, and precipitation ($P > 0$ always) leads to NEGATIVE $\partial S/\partial z$. 
References

Baumert et al., 2005:
Gill, A., 1982: Atmospheric & Oceanic Dynamics