II.

THE STABILITY OR INSTABILITY OF THE STEADY MOTIONS OF A PERFECT LIQUID AND OF A VISCOUS LIQUID. PART I.: A PERFECT LIQUID.

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INTRODUCTION AND SUMMARY OF CONTENTS.

It is a well-known experimental fact that when a liquid of small viscosity, such as water, flows through a straight circular pipe under applied pressure or under the action of gravity, the steady motion—in which, of course, each particle describes a straight line—may be unstable. The subject has been investigated experimentally by Osborne Reynolds,¹ who found that the motion is stable so long as the mean velocity does not exceed a certain limit depending on the radius of the pipe and on the nature of the liquid. This limit, beyond which instability sets in and the motion becomes turbulent, he found to vary directly as the kinematic viscosity, and inversely as the radius of the pipe—results to which he was led also by considerations of the theory of dimensions.

The question has also been attacked theoretically, chiefly by Lord Rayleigh, Lord Kelvin, and Reynolds himself. Lord Rayleigh² has ignored the effect of viscosity in the disturbed motion—a simplification which renders such problems much more amenable to mathematical treatment. One series of his papers deals with flow in plain strata between fixed parallel walls; and he arrives at the conclusion that the motion is not unstable, provided the law of flow is such that the velocity-gradient continually increases or continually decreases (algebraically) from one wall to the other. Quoting his own words,³ "To be more precise, it was proved that if the deviation from the regularly

¹ "An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels," Phil. Trans., t. clxxiv., Part III., p. 935 (1883); Sc. Papers, t. II., p. 51.
² Detailed references are given in the text.

stratified motion were, as a function of the time, proportional to \( t^m \), then \( n \) could have no imaginary part." In the paper quoted from he discusses flow in cylindrical layers, as through a straight circular pipe; and, as a particular case of a more general result, he concludes that when the distribution of velocity is that which actually exists in the case of a viscous liquid, the steady motion is not unstable. He considers that case also of flow in cylindrical layers in which the particles describe circles about a common axis, and concludes that the motion is stable if the rotation either continually increases or continually decreases in passing outwards from the axis. This condition is satisfied if the law of velocity is that which obtains in a viscous liquid between long concentric cylinders of which one is fixed and the other made to rotate. It has been found experimentally by Mallock\(^1\) and by Couette\(^2\) that, under these circumstances, the motion of water is unstable if the velocity be sufficiently great.

Accordingly, in the second and third of the three classes of motion referred to, the behaviour of natural liquids, as tested by Reynolds, Mallock, and Couette, appears to differ from that attributed to perfect liquids by Lord Rayleigh. (I am not aware that any experiments have been made dealing directly with the first class of motions, that in plane layers.) There is thus a difficulty in reconciling theory and experiment.

Portions of Lord Rayleigh's argument have, however, been criticised adversely by Lord Kelvin\(^3\) and by Love.\(^4\)

When viscosity is taken into account, the mathematical difficulties involved in a discussion of the question of stability are much greater. Lord Kelvin\(^5\) has attacked the question under such conditions. He has considered two problems of motion in plane layers—one that of a liquid undergoing shear at a uniform rate, the other that of a liquid flowing between two fixed parallel planes—and concludes that in each case the motion is stable for sufficiently small disturbances, but that for disturbances exceeding a certain magnitude the motion becomes unstable, and that this limiting magnitude is smaller the smaller the viscosity—a view to which Reynolds has been led by his experiments. His mode of solving the latter problem applies equally to the former, as he points out; but these solutions have been rejected by Lord Rayleigh. Lord Kelvin has also given another solution of the former problem which Lord Rayleigh regards as satisfactory.

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2. Annales de Chimie et de Physique [6], 21, p. 433.
Reynolds also has attacked theoretically the latter of the two problems discussed by Lord Kelvin, and has obtained an inferior limit to the velocity for which the motion can be unstable; his result is of the same order of magnitude as that which he obtained experimentally in the somewhat different case of flow through a pipe.

An inferior limit, different from that of Reynolds, but of the same order of magnitude, has been obtained theoretically by Sharpe, who has also deduced, in the case of flow through a pipe, a limit of the order of that observed by Reynolds.

In the case of the liquid shearing uniformly, H. A. Lorentz has obtained a limit which is of the same order.

Both these writers use Reynolds' method.

The investigation here presented deals exclusively with questions in which viscosity is altogether ignored.

Its contents may be summarized as follows:

In Chapter I, pp. 17-42, cases of motion in plane strata are discussed.

In Art. 1, p. 17, a brief outline is given of Lord Rayleigh's investigation of the fundamental free disturbances, reference being made to Lord Kelvin's objection, which I confess I do not understand, and to Lord Rayleigh's reply thereto.

In Art. 2, p. 20, I have given what appear to be the most important portions of Love's criticism of these investigations, and have remarked upon them in Art. 3, p. 22. In brief, Professor Love has made three objections to Lord Rayleigh's solution, viz.: (1) the free disturbances involve slipping in the interior of the fluid; (2) the wave-velocity is restricted within certain limits; (3) it has not been shown that an arbitrary disturbance can be replaced by a system of Lord Rayleigh's type. Of these it appears to me that (1) and (2) have no force whatever, but that (3) calls for further examination.

In Art. 3a, p. 23, I point out, however, that in Lord Rayleigh's free disturbances, although the velocity at a given point, as given by terms of the first order of small quantities, is periodic in time, yet the amplitude of the waves generally increases; owing to this, in itself, his conclusion as to stability may require modification; but this cannot be decided without taking into account terms of smaller order.

In Art. 4, p. 24, Professor Love's third objection is considered; and taking

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3 Abhandlungen über theoretische Physik, Band 1., s. 322.
the simplest possible case, that of a liquid which is shearing uniformly, it is shown that, at least in this case, the most general disturbance can be resolved into a series of the type obtained by Lord Rayleigh. The resolution is effected for an example of the simplest type analytically, i.e., one in which the initial velocity-components are sine-cosine functions of the coordinates; and when the initial disturbance is of this character an expression is obtained for the velocity at right angles to the bounding-planes at any time.

In Art. 5, p. 26, this same result is obtained more directly from the fundamental equations without reference to Lord Rayleigh's "free modes." When the disturbance is three-dimensioned, the expressions for the velocities parallel to the bounding-planes involve transcendental integrals, and accordingly the complete solution is given for the two-dimensioned case only.

The solution thus obtained is periodic in the direction of flow and of assigned wave-length; in Arts. 6, 7, pp. 28, 29, it is indicated how the solution is to be modified in two other instances in which other and more definite conditions are to be satisfied at the ends of the stream.

In Art. 8, p. 29, the solution which has been obtained is examined; and it is readily seen that if the initial wave-length perpendicular to the bounding-planes is small compared with the wave-lengths in the directions parallel to them, and also small compared with the distance between them, the original disturbance increases and attains a maximum value, much greater than its initial, at a certain critical time, after which it diminishes without limit. For the two-dimensioned case, the order of the increase can be stated in a simple form in two extreme cases:—if the wave-length in the direction of flow is large compared with the thickness of the stream, the ratio in which the kinetic energy of the relative motion increases is of the order of the square of the number of wave-lengths perpendicular to the stream which are contained in the original disturbance; while if the wave-length in the direction of flow is small compared with the thickness of the stream, the ratio of increase is of the order of the square of the ratio of the wave-length in the direction of flow to that perpendicular to the boundaries. This constitutes, I think, a satisfactory explanation of the instability which observations of motion in pipes lead us to expect also in cases of plane stratified flow.

In Art. 9, p. 32, it is pointed out that coexistence of the stability or neutrality, established by Lord Rayleigh, in the case of each of the fundamental modes of disturbance, with what may, I think, be described as practical instability for others of a more general type is quite in keeping with the teaching of Fourier analysis; that the question of the stability of a state of equilibrium is in reality decided by a potential-energy criterion; and that the light thrown on the question by a knowledge of the reality of the "free periods" is only
indirect. The case of a system possessing only two coordinates is considered; and it is shown that if there is no potential-energy function, stability, or rather neutrality, of the two fundamental modes is quite consistent with very narrow limits of stability for a combination of both. When the question is one of the stability of a state of motion, it does not appear to have been established, for a system possessing an infinite number of coordinates, that reality of the periods of the fundamental disturbances necessitates stability for an arbitrary disturbance, however small, even when there is a potential-energy function. A concrete instance—that of an unsymmetrical spinning-top standing up and acted on by gravity—is given, in which the reality of the two fundamental periods is compatible with practical instability for a more general disturbance. It seems only another mode of contrasting these cases to assert that equality of two periods cannot affect the stability of equilibrium of a system possessing an energy-function; but that equality of periods may destroy, and approximate equality may endanger, the stability of a state of motion, and that, moreover, the extent of the danger cannot be judged by a mere comparison of the periods.

In Art. 10, p. 36, it is pointed out how the impossibility of inferring stability in general from that of the fundamental disturbances is connected with the fact that the latter do not possess the property characteristic of the oscillations about a state of equilibrium of a system having a potential-energy function, viz.:—that the integrated product of the corresponding velocities in any two principal modes vanishes.

In Art. 11, p. 37, it is shown from the solution obtained that if the end-conditions are such that the velocity components are periodic in the direction of flow, the energy of the actual as well as of the relative motion increases for a time, and that this arises from work being done by the pressures, which cannot be strictly periodic in the direction of flow.

In Art. 12, p. 38, it is shown that any disturbance of an ordinary type must remain finite, and that in the most general one, provided the velocities possess a definite wave-length in the direction of flow, the relative velocity component in that direction, as determined by the solution given, eventually diminishes indefinitely, varying inversely as the time, while the component at right angles eventually varies inversely as the square of the time, so that it may be said the steady motion is stable, provided the initial disturbance is small enough.

Art. 13, p. 39, deals briefly with the more general case of a stream composed of a number of plane layers, each of which is shearing uniformly, but at a rate which is different in different layers. The solution of even the two-dimensioned problem cannot readily be given in a form which admits of
quantitative comparisons; but it is shown that here, too, Lord Rayleigh's analysis suffices to include the most general disturbance, and that some disturbances of initially simple type will increase very much.

The chapter concludes with a brief consideration, in Art. 14, p. 41, of the case in which, in the steady motion, the rate of shearing varies continuously from one bounding-plane to the other, instead of by abrupt changes. Mathematical difficulties render this portion of the discussion very unsatisfactory; but reasons are put forward for holding that at any rate if a disturbance has a wave-length in the direction of flow which is sufficiently short, and has initially one in the perpendicular direction which is much shorter, it will increase very much (and afterwards diminish indefinitely).

Chap. II., pp. 43-60, deals with flow in cylindrical strata, through a pipe whose section is a circle, or an annulus between two concentric circles.

Art. 15, p. 43, contains a brief account of Lord Rayleigh's discussion of the fundamental free modes of disturbance. The only case in which he has actually obtained the solution is that in which the law of velocity in the steady motion is that appropriate to a viscous liquid in a complete circular pipe, and then only for disturbances symmetrical about the axis. This is the only law of flow, and this the only type of disturbance, which are at all tractable; and the remainder of the chapter is accordingly devoted to the consideration of this particular problem. As in the case of plane strata, discussed in Chapter I., each fundamental mode involves slipping in the interior and, of course, at the boundaries.

In Art. 16, p. 44, it is shown how any symmetrical disturbance may be resolved into a system of Lord Rayleigh's type.

And in Art. 17, p. 46, the result to which this leads is obtained directly from the fundamental equations.

In Art. 18, p. 47, the solution is written down for an initial disturbance of type analytically simple, the radial velocity being \( \sin m (r - b) \sin ka \), \( b \) being the inner radius (which may be zero), and \( z \) being measured in the direction of flow; this solution is in terms of somewhat complicated integrals involving Bessel functions of a purely imaginary argument. The approximate values of these integrals, under certain conditions, are examined with a view to find the magnitude of the disturbance at subsequent times; and, in the definite case in which the wave-length in the direction of flow is small compared with the distance of the point considered from the axis, it is shown that if the initial wave-length radially is still much smaller as a certain critical time is approached, the disturbance increases in a very great ratio if the point be not near a boundary. For any point, this critical time depends on its distance from the axis. The justification which it has been thought
desirable to give of the approximations used renders this and some succeeding portions of the discussion somewhat tedious.

In Art. 19, p. 53, it is shown that this initial disturbance, and any other in which the velocities have a definite wave-length in the direction of flow, must eventually diminish indefinitely according to the same laws as in the plane stratified case.

In Art. 20, p. 54, another instance of initial disturbance is considered in which the radial velocity is \( \sin m (r^2 - \theta^2) \sin hx \). The wave-length along the pipe is supposed small compared with the outer radius, a result similar to that deduced for the former example being obtained. In this case, however, the critical time is the same at all points; and accordingly an approximate expression is obtained for the ratio of increase of the energy of the relative motion throughout the whole pipe at this critical time.

In Art. 21, p. 58, the initial disturbance of the preceding Article is discussed under a different extreme supposition, viz., that the wave-length along the pipe is large compared with the outer radius; and similar conclusions are drawn.

Although quantitative comparison is easier in the extreme cases of waves which are long and of waves which are short in the direction of flow, there is reason to suppose that a disturbance of any wave-length whatever in this direction, if of much shorter, and sufficiently short, wave-length in the direction at right angles, will increase very much, provided equations remain valid in which the squares of small quantities are neglected.

Chapter III., pp. 61-68, discusses steady motion in cylindrical strata, rotating round a common axis.

Art. 22, p. 61, deals with Lord Rayleigh's brief reference to this case.

The analysis appropriate to the investigation of the two-dimensioned disturbances which are harmonic functions of the time is given in Art. 23, p. 61. It is seen that the only law of flow for which the solution can readily be obtained is that obeyed by a viscous liquid when one or both of the cylindrical boundaries are made to rotate. The solution again involves slipping in the interior as well as at the boundaries. It is shown how the general two-dimensioned disturbance can be propagated by means of elementary ones of the type obtained, and how the result to which this resolution leads may be obtained directly, without reference to the fundamental free modes.

In Art. 24, p. 63, it is shown that any two-dimensioned disturbance, in which initially the relative velocity components vary as \( \cos s \theta \), \( \sin s \theta \), \( s \) being a definite number, will eventually diminish indefinitely according to laws similar to those which hold for the cases discussed in the preceding chapters.
In Art. 25, p. 64, there is traced to some extent the history of a disturbance whose initial type is so chosen as to make the analysis as simple as possible, viz., one for which the stream-function is \( \sin c^2 (r^2 - b^2) \sin \theta \), \( b \) being the inner radius. Only the case of one definite alternative of the relative magnitudes is fully discussed, the choice being made so as to obtain a problem sensibly different from the principal one of Chapter I. It appears that if \( c \) is sufficiently large, the disturbance will increase very much before dying out. The critical time is the same for all points, and an approximate expression is obtained for the ratio in which the kinetic energy of the relative motion throughout the whole liquid is increased at this critical time.

One case constitutes an exception to these statements. If in the steady motion the liquid rotates as a rigid body, then any small disturbance, as far as terms of the first order show, neither increases nor decreases, but is simply carried round with the liquid.

It is held that as far as this investigation goes no contradiction between theory and experiment is revealed. The apparent paradox that the motion of a liquid devoid of viscosity, if such existed, would be stable, while that of an actual liquid of small viscosity is found by experiment to be highly unstable, is disposed of by showing that though the perfect liquid may be said to be stable if the disturbance is small enough, yet the limit of stability, or, to be accurate, the limit within which it is legitimate to rely on equations which take account of only the first powers of small quantities, depends on the nature of the disturbance, and may be diminished indefinitely by a suitable choice. And the opinions expressed by Lord Kelvin and by Reynolds, that the limit of stability of flow of a viscous liquid diminishes indefinitely with the viscosity, are to some extent confirmed. Any further remarks on the effect of viscosity are postponed.

It seems worthy of note that, as I understand it, the instability which is actually observed in these cases may be described as a disturbance periodic in time and increasing with the distance travelled by the particles rather than as one periodic in distance, and increasing with the time; that the disturbances are "forced" rather than "free." I am not clear as to how far the problems are analytically identical.

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\[ \text{Professor Love has reminded me of this distinction.} \]
CHAPTER I.

RECTILINEAR MOTION IN PLANE LAYERS, CHIEFLY THE CASE OF A LIQUID SHEARING UNIFORMLY.

ART. 1. Lord Rayleigh’s Investigations.

The oscillations which are possible in a stream of liquid, supposed frictionless, flowing between two fixed parallel planes, have been discussed in a series of papers by Lord Rayleigh. It appears desirable to give a brief account of some of his investigations. In one of his earliest papers on the subject,* he supposes that the axis of y is drawn at right angles to these planes, and that the velocity in the steady motion is \( U \) in the direction of the axis of x, \( U \) being a function of y only. He considers only two-dimensioned disturbances; in these denote the x, y components of velocity by \( U + u, v \); let \( Z \) denote the vorticity in the steady motion, i.e. \( \frac{1}{2} \frac{dU}{dy} \), and \( \zeta \) denote the additional vorticity, i.e. \( \frac{1}{2} (\frac{du}{dy} - \frac{dv}{dx}) \). Since, in the absence of friction, the vorticity of each element remains constant, we have

\[
\frac{d}{dt} (Z + \zeta) + (U + u) \frac{d}{dx} (Z + \zeta) + v \frac{d}{dy} (Z + \zeta) = 0,
\]

or, if we retain only the first powers of small quantities,

\[
\frac{d\zeta}{dt} + U \frac{d\zeta}{dx} + v \frac{d\zeta}{dy} = 0,
\]

which may be written in the form

\[
\left( \frac{d}{dt} + U \frac{d}{dx} \right) \left( \frac{du}{dy} - \frac{dv}{dx} \right) + \psi \frac{d^2 U}{dy^2} = 0.
\]

Introducing the supposition that as functions of x, u and v vary as \( e^{ikx} \), and using the equation of continuity

\[
\frac{du}{dx} + \frac{dv}{dy} = 0,
\]

or, as it now becomes,

\[
iku + dv/dy = 0,
\]

we obtain, on elimination of \( u \),

\[
\left( \frac{d}{dt} + ikU \right) (dx/dy - k^2 v) - ikv dx/dy^2 = 0.
\]


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If we further suppose that, as a function of $t$, $v$ is proportional to $e^{nt}$, where $n$ is a real or complex constant this becomes

$$(n + kU)(d^2v/dy^2 - k^2v) - kvUdy^2 = 0.$$  \(7\)

Lord Rayleigh devotes special attention to various cases in which the stream is composed of several separate layers in each of which the rotation in the steady motion is constant, but a different constant for different layers. He regards $U$ as continuous, some cases in which $U$ is supposed discontinuous having been discussed by him in a previous paper "On the Instability of Jets."*

If, in any layer, the rotation $Z$ is constant,

$$dU/dy^2 = 0.$$  \(8\)

The solution of this is

$$v = Ae^{xy} + Be^{-xy},$$  \(9\)

where $A$ and $B$ are constants, real or complex. For each layer of constant $Z$, a fresh solution with different constants is to be taken, the partial solutions being fitted together by means of the proper conditions at the surfaces of transition. One of these conditions is

$$\Delta v = 0.$$  \(10\)

Another is obtained by integrating (7) across the surface of transition, and is

$$(n + kU) \Delta \frac{dv}{dy} - kvU \Delta \frac{dU}{dy} = 0.$$  \(11\)

This last equation, to be satisfied at the fixed plane which is the separating surface in the steady motion, expresses the condition that there shall be no slipping at a surface of transition. At first sight it might appear that this condition requires

$$\Delta (U + u) = 0.$$  \(12\)

at the fixed plane in question. What is required, however, is that (12) should be satisfied at the disturbed surface; and it may be shown that this reduces to (11). This may be seen as follows:—Let the surface of separation be

$$F = y - h \cos (nt + kx) = 0,$$

and suppose on the positive side

$$U + u = U + 2Zy + (Ae^{xy} - Be^{-xy}) \cos (nt + kx),$$
$$v = (Ae^{xy} + Be^{-xy}) \sin (nt + kx),$$

and on the negative

$$U + u = U + 2Zy + (A'e^{xy} - B'e^{-xy}) \cos (nt + kx),$$
$$v = (A'e^{xy} + B'e^{-xy}) \sin (nt + kx).$$

* P. I. M. S. x., p. 4, 1878; Scientific Papers, t. i., p. 361.
Neglecting, of course, terms of higher order than the first power of small quantities, the condition for no slipping at the separating surface, obtained by equating the two values of \( u \) at the surface in question and dividing by \( \cos (nt + kx) \), becomes

\[
2h \Delta Z + \Delta (A - B) = 0.
\]

In virtue of the relation

\[
\frac{dF}{dt} + (U + u) \frac{dF}{dx} + v \frac{dF}{dy} = 0, *
\]

we have

\[
(n + kU) h \sin (nt + kx) + v = 0,
\]

and eliminating \( h \), the result follows.

In cases where \( \frac{d^3U}{dy^3} = 0 \), the substitution of (8) for (7) or the equivalent supposition that the vorticity is unchanged,† constitutes a limitation on the disturbance. In order to obtain a general solution we must retain the factor \( n + kU \) in (7). For any value of \( y \) which makes \( n + kU = 0 \) (8) need not be satisfied; and thus any value of \( -kU \) is an admissible value of \( n \) satisfying all the conditions of the problem. Such a solution involves slipping between layers whose separating surface in the steady motion is given by the value of \( y \) referred to.

Moreover, as this separating surface may equally well be a surface separating layers of different rotation in the steady motion, we may have solutions in which (11) is violated if \( n + kU = 0 \) at the surface. If there be no slipping at a separating surface for which \( n + kU = 0 \), equation (11), as Lord Rayleigh points out, reduces to \( v = 0 \).

Lord Rayleigh then proceeds to consider the case in which \( \frac{d^3U}{dy^3} \) is not zero, and shows that if it be one-signed throughout, no complex value of \( n \) can occur, and concludes that, if this condition be satisfied, the motion is thoroughly stable.

Lord Kelvin has argued‡ that when, in (7), \( n + kU = 0 \), there is a "disturbing infinity which vitiates the seeming proof of stability contained in Lord Rayleigh's equations."

I do not understand clearly what Lord Kelvin's objection really is; possibly he contends that where \( n + kU = 0 \), equation (7) when written in the form

\[
\frac{d^3v}{dy^3} - \frac{\partial v}{n + kU} = \frac{kud^3U}{n + kU} \tag{13}
\]

gives an infinite value for \( \frac{d^3v}{dy^3} \), or that the slipping to which Lord Rayleigh's solution leads renders the motion unstable.

† Equation (8) is equivalent to \( \frac{d^3v}{dx^3} + \frac{d^3v}{dy^3} = 0 \) or \( d/\delta (\nu/\delta / \delta - \delta v/\delta y) = 0 \); thus we have \( d\nu/\delta x - d\nu/\delta y = f (y, t) \), and this function of \( y, t \), necessarily vanishes since the velocities are periodic in \( x \).
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In a later paper, which refers chiefly to motion through a circular pipe, Lord Rayleigh points out that, if \( n \) be complex, there is no "disturbing infinity," and that therefore his argument does not fail if regarded as one for excluding complex values of \( n \), though what happens when \( n \) has a value such that \( n + kU \) vanishes at an internal point, is a subject for further consideration.*

To this subject he returns; and both in the case in which the vorticity in the steady motion is constant through certain layers, but discontinuous at their boundaries, and that in which it is continuous throughout but varying, he concludes that the infinities which present themselves when \( n + kU \) is zero, do not seriously interfere with the application of the general theory, so long as the square of the disturbance from steady motion is neglected.†

And, in his latest paper on the subject, taking the simple case in which in the steady motion the velocity increases uniformly from each wall to the centre of the stream, he has examined the effect of including in the investigation the squares and higher powers of the small quantities as far as the fifth power. He concludes that there is no sign of the amplitude of a wave tending spontaneously to increase, as far as his investigation goes.‡ His discussion is, however, limited to the very restricted class of disturbances which do not involve any slipping at the surface where the vorticity is discontinuous. And if such slipping be introduced, the contrary result would apparently be arrived at. (See Art. 33 above.)

**ART. 2. Prof. Love’s Criticism of the above.**

In a criticism of these investigations of Lord Rayleigh, Professor Love writes§ [having replaced \( n/k \) by \( -V \), so that equation (7) becomes

\[
(U - V)(d^2v/dy^2 - k^2v) - v\partial^2 U/dy^2 = 0 \tag{14}
\]

"In order that the disturbance may be propagated by waves in the manner supposed, it must be possible to assign a real quantity \( V \) so that a function \( v \) may exist which (i) satisfies the differential equation [14, above] for all values of \( y \) in a certain real interval, (ii) vanishes at the limits of this interval, (iii) is finite, and has a finite and continuous differential coefficient

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‡ "On the Propagation of Waves upon the Plane Surfaces separating two Portions of Fluid of Different Vorticities," P. L. M. S., xxvii., 1895; Collected Papers, iv.—the concluding sentence.
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in this interval. Further, in order that the method may apply to an arbitrary initial disturbance, it is necessary that there should be a series of such quantities $V_r$, and, associated with each, a function $v_r$ of such a character that an arbitrary function of $y$ can be expanded in a series of the form

$$\sum A_r v_r,$$

which converges in the given interval. The quantities $V_r$ are required to exist for all real values of $k$.

"Lord Rayleigh has proved that it is impossible to satisfy the differential equation and the boundary conditions with a complex value of $V$, if $d^2U/dy^2$ is one-signed between the boundaries; and he concluded that, under this condition, the steady motion expressed by $U$ must be stable. It appears, however, that this conclusion required additional justification, inasmuch as there is no evidence to show that every disturbance will be propagated by waves in the manner supposed. Lord Rayleigh has further remarked that it is impossible to satisfy the differential equation and the boundary conditions with any value of $V$ for which $U - V$ and $d^2U/dy^2$ have the same sign everywhere between the boundaries."

Professor Love then proceeds to examine a certain example in which $d^2U/dy^2$ is one-signed between the boundaries, and proves that in its case,* "though there may be a finite number of values of $V$ for which the differential equation

$$(U - V)(d^2v/\frac{dy^2}{\frac{k^2v}}) = v \frac{d^2U}{dy^2}$$

has a solution $v$, which vanishes when $y = h_1$ and when $y = h_2$, there cannot be an indefinite series of such values. It follows that, though there may be particular types of disturbance which can be propagated by wave-motion in the manner supposed, this cannot be true for a general disturbance."

Further on Professor Love refers to the case in which $U$ is a linear function of $y$: he writes†:

"The differential equation becomes

$$\frac{d^2v}{dy^2} - k^2v = 0,$$

and a solution vanishing when $y = h_1$ is

$$v = A \sinh k(y - h_1),$$

but we cannot make it vanish also when $y = h_2$. In this case it has been suggested that a possible wave-motion might be found by taking $V$ equal to the value of $U$ at one line, $y = \alpha$ say, between $h_1$ and $h_2$. Then the

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* L. c., p. 207.
† L. c., p. 212.
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differential equation [(15), above] need not be satisfied when \( y = a \). We should then have to take

\[
\begin{align*}
v &= A \sinh k (y - h_1), \quad a > y > h_1, \\
v &= B \sinh k (h_2 - y), \quad h_2 > y > a.
\end{align*}
\]

To make \( v \) and \( dv/dy \) continuous at \( y = a \), we should require

\[
\begin{align*}
A \sinh k (a - h_1) &= B \sinh k (h_2 - a), \\
A \cosh k (a - h_1) &= -B \cosh k (h_2 - a),
\end{align*}
\]

and these cannot be satisfied when \( h_1 \) is different from \( h_2 \). Thus there would be in this case no disturbance which could be propagated by waves in the manner supposed. Yet the example afforded by initial disturbances

\[
\begin{align*}
u &= C (2y - h_1 - h_2) \cos kw, \\
v &= kC (y - h_1) (y - h_2) \sin kw,
\end{align*}
\]

shows that some varied motion is possible which initially is periodic in \( x \) with given wave-length. Lord Rayleigh's method does not avail for the discovery of this motion, nor for determining whether the original steady motion is stable for this type of disturbance."

And in the introduction to his paper he expresses the opinion that* "the general conclusion seems to be that wave-motions of Lord Rayleigh's type can only occur in some very special cases, and that his method does not avail for the determination of a criterion of stability when the disturbance is of a general character."

**Art. 3. Remarks on Love's Criticism.**

It appears to me that the remarks which I have quoted embody two misconceptions, and that as a consequence the mathematical investigations in Professor Love's paper are in great measure irrelevant.

In the first place, we are not entitled \textit{a priori} to impose the condition that in a perfect fluid \( dv/dy \) is continuous across a plane parallel to \( y \). This condition is equivalent to requiring \( du/dx \) to be continuous, and therefore either that \( u \) is continuous, or that any discontinuity in it is independent of \( x \): it involves then either that there is no slipping, or that there is some restriction on its amount; but we cannot control slipping in a perfect fluid. Whenever the continuity of \( dv/dy \) is secured, I apprehend it is by the integration of equation (7) across the plane in question, as Lord Rayleigh has stated, and thus, wherever \( n + kU \) or \( n (V - U) \) is zero, discontinuity is permissible.

Again, we have no right to say that the possible values of \( V \) or \( -n/k \)

* L. c., p. 199.
should be unrestricted in magnitude or infinite in number (or, on the other hand, to impose any restriction in either of these respects). The oscillations characteristic of a compressible fluid, for instance, are propagated with one unique velocity, or more properly with only two velocities equal in magnitude and opposite in sign.

There is more force in the objection that Lord Rayleigh has not proved that an arbitrary disturbance can be propagated in the manner he supposes. He has, however, made out a prima facie case. And a satisfactory investigation of the possibility of the expansion of an arbitrary function in a series of given functions, as by Fourier's series, is generally a matter of difficulty. In the case in which the stream is composed of layers of constant vorticity, it may be proved that the requisite expansion is always possible, when the arbitrary function is of an ordinary character.

Art. 3a. The Wave-Amplitude generally increases.

There is, however, a circumstance connected with any fundamental free disturbance which to some extent should modify Lord Rayleigh's conclusion that for such a disturbance the steady motion is stable. Lord Rayleigh has shown that, in a stream moving with uniform velocity, if a wavy surface of discontinuity be created parallel to the direction of motion, and slipping occurs in the direction of flow, the amplitude of the waves increases* (as illustrated by the flapping of sails and flags). It may be shown that this is the case also in each free disturbance of the fluid shearing. If, for simplicity, the bounding planes be supposed at an infinite distance from the surface of discontinuity taken to coincide (approximately) with the plane \( y = 0 \), we have, on the positive side of this surface,

\[
v = Ae^{ky} \sin k(x - Ut),
\]

\[
U + u = U + 2Zy - Ae^{ky} \cos k(x - Ut).
\]

If \( y = f(x, t) \) be the actual surface of separation (accurate to terms of the first order),

\[
df/dt + Udf/dx = v = A \sin k(x - Ut); 
\]

and the general solution of this, which is of wave-length \( 2\pi/k \) in \( x \), is

\[
f = At \sin k(x - Ut) + C \cos k(x - Ut) + C' \sin k(x - Ut).
\]

This increase of amplitude, moreover, occurs in most of the more general cases of flow discussed by Lord Rayleigh—in that alluded to in the concluding paragraph of Art. 1, if slipping be set up; that of Art. 13, below; that of Chap. II.; that of Chap. III.

* "On the Instability of Jets," Proc. L. M. S., x., p. 4, 1879; Scientific Papers, i., p. 367
ART. 4. Arbitrary Disturbance in uniformly shearing Liquid resolved into a series of Lord Rayleigh's type. Case where initial velocities are sine-cosine functions of coordinates.

I proceed then, in the simplest possible case, that in which the velocity in the steady motion is, from one boundary to the other, a linear function of y, to consider the expansion of an arbitrary function of y in terms of the functions which present themselves in Lord Rayleigh's investigation, and to examine the propagation of an arbitrary disturbance. If the fixed boundaries be denoted by \( y = 0, \ y = b \), the problem in expansions is as follows:—

Given an arbitrary function \( f(y) \), to find a function \( \phi \), such that for values of \( y \) between 0 and \( b \)

\[
f(y) = \int_0^b \phi(\eta) F_\eta(y) \, d\eta \tag{16}
\]

where \( F_\eta(y) \) is a given function of \( y \) defined in the following manner:—

When \( y \) is less than \( \eta \), \( F_\eta(y) = A \sinh \lambda y \),
when \( y \) is greater than \( \eta \), \( F_\eta(y) = B \sinh \lambda (b - y) \),

\( A, B \), being connected by the relation

\[
A \sinh \lambda \eta = B \sinh \lambda (b - \eta),
\]

and \( \lambda \) being given. As we may take any convenient multiple of each function, we will choose

\[
A = 1/\sinh \lambda \eta, \quad B = 1/\sinh \lambda (b - \eta).
\]

We notice that these functions of \( y \) do not conform to the relation which exists among the normal coordinates of a conservative system oscillating about a position of equilibrium, viz.:

\[
\int_0^b F_{\eta_1}(y) F_{\eta_2}(y) \, dy = 0. \tag{17}
\]

With the above values of \( A, B \), equation (16), if it exists, assumes the form

\[
f(y) = \int_0^y \phi(\eta) \sinh \lambda (b - \eta) \sinh \lambda (b - y) \, d\eta + \int_y^b \phi(\eta) \sinh \lambda \eta \sinh \lambda (b - y) \, d\eta. \tag{18}
\]

By differentiation, we obtain

\[
f'(y) = -\lambda \int_0^y \phi(\eta) \cosh \lambda (b - \eta) \sinh \lambda (b - y) \, d\eta + \lambda \int_y^b \phi(\eta) \cosh \lambda \eta \sinh \lambda (b - y) \, d\eta, \tag{19}
\]

\[
f''(y) = \lambda^2 \int_0^y \phi(\eta) \sinh \lambda (b - \eta) \cosh \lambda (b - y) \sinh \lambda (b - \eta) \, d\eta + \lambda^2 \int_y^b \phi(\eta) \sinh \lambda \eta \cosh \lambda (b - y) \sinh \lambda \eta \, d\eta - \frac{\lambda \phi(y) \sinh \lambda b}{\sinh \lambda (b - y) \sinh \lambda y}, \tag{20}
\]

and hence

\[
f''(y) = \lambda^2 f(y) - \frac{\lambda \phi(y) \sinh \lambda b}{\sinh \lambda (b - y) \sinh \lambda y} \tag{21}
\]

giving

\[
\phi(y) = \frac{\lambda^2 f(y) - f''(y)}{\lambda \sinh \lambda b} \sinh \lambda (b - y) \sinh \lambda y; \tag{21}
\]
and (16) thus assumes the form
\[
\lambda \sinh \lambda b f(y) = \sinh \lambda (b - y) \int_{y}^{\eta} \sinh \lambda \eta \{\lambda^2 f(\eta) - f''(\eta)\} d\eta
\]
\[+ \sinh \lambda y \int_{0}^{\eta} \sinh \lambda (b - \eta) \{\lambda^2 f(\eta) - f''(\eta)\} d\eta; \tag{22}\]
and it may now be directly verified that this result is true, provided \(f(y)\) and \(f'(y)\) are finite, continuous, and differentiable between 0 and \(b\), and \(f(0)\) and \(f(b)\) both vanish, as is the case in the problem to which the theorem is to be applied. If \(f(0), f(b)\) are not zero, we require to add to the right-hand member
\[
\lambda f(0) \sinh \lambda (b - y) - \lambda f(b) \sinh \lambda y;
\]
and even this apparent exception may be made to conform to (16), if we agree to consider that \(\phi(\eta)\) becomes infinite at the limits \(0, b\), in such a fashion that for infinitesimal ranges \(d\eta\) at the lower limit \(\phi(\eta)d\eta = f(0)/\sinh \lambda b\), and at the upper, \(\phi(\eta)d\eta = -f(b)/\sinh \lambda b\).

If, then, there be an initial disturbance in which \(v = f(y) e^{\lambda x}\), its value at the time \(t\) as thus obtained is given by
\[
\lambda \sinh \lambda b v = \sinh \lambda (b - y) \int_{y}^{\eta} \sinh \lambda \eta \{\lambda^2 f(\eta) - f''(\eta)\} e^{\lambda(x-\eta t)} d\eta
\]
\[+ \sinh \lambda y \int_{0}^{\eta} \sinh \lambda (b - \eta) \{\lambda^2 f(\eta) - f''(\eta)\} e^{\lambda(x-\eta t)} d\eta, \tag{23}\]
in which \(U\) is a linear function of \(\eta\).

This is the solution on the supposition that the disturbance continues to have a wave-length in the \(x\) direction equal to \(2\pi/\lambda\).

The discussion may be made more general by extending its scope so as to include three-dimensional disturbances at least as far as finding the value of \(v\). We may, without loss of generality, suppose that one of the bounding planes is reduced to rest, as any other case may be obtained from this by replacing \(x\) by \(x - ct\), where \(c\) is constant. Let then the velocity in the steady motion be given by \(U = By\). Consider the propagation of the disturbance in which the initial values of \(u, v, w\) are
\[
\begin{align*}
u_0 &= A \sin lx \cos my \cos nz, \\
v_0 &= B \cos lx \sin my \cos nz, \\
w_0 &= C \cos lx \cos my \sin nz, \tag{24}\end{align*}
\]
where \(\sin mnb\) is zero, and, as follows from the equation of continuity,
\[
LA + mB + nC = 0. \tag{25}\]
In each “free” disturbance, \(v\), as a function of \(x, y, t\), is to be taken to vary as \(\sinh \lambda y e^{\eta(x-\eta t)}\) on one side of the plane of discontinuity \(y = \eta\), and as \(\sinh \lambda (b - y) e^{\eta(x-\eta t)}\) on the other, where now
\[
\lambda^2 = b^2 + n^2, \tag{26}\]

* The solution can be made to satisfy definite assigned end-conditions: see below, Arts. 6, 7.
instead of \( \lambda^2 = \nu \) as in the two-dimensioned disturbances considered by Lord Rayleigh. The initial value of \( \nu \) given by (24) is the real part of \( Be^{i\pi} \sin m \eta \cos nx \); and when this complex expression is expanded, as far as it involves \( x \) and \( y \), by the aid of (22) in the form

\[
\int_0^\infty F_\eta(y) \phi(\eta) d\eta e^{i\sigma t},
\]

the corresponding value at any time \( t \) is obtained by multiplying each element of this integral by \( e^{-i\sigma t} \). Thus, on rejecting the imaginary parts, we obtain a value for \( \nu \) given by the equation

\[
\lambda \sinh \lambda \nu \eta \frac{B \cos nx}{(\lambda^2 + m^2) B \cos nx} = \sinh \lambda (b - y) \int_0^\nu (\lambda^2 + m^2) \sinh \lambda \eta \sin m \eta \cos l(x - \beta \eta) d\eta
\]

\[
+ \sinh \lambda y \int_y^\nu (\lambda^2 + m^2) \sinh \lambda (b - \eta) \sin m \eta \cos l(x - \beta \eta) d\eta.
\]

(27)

On performing the integrations this result is seen to be equivalent to

\[
\frac{2 \nu \sinh \lambda \nu}{(\lambda^2 + m^2) B \cos nx} = \frac{\sinh \lambda \nu \{lx + (m - i\beta \eta) y\} - \sinh \lambda (b - y) \sin lx - \sinh \lambda \nu \{lx + (m - i\beta \eta) b\}}{\lambda^2 + (m - i\beta \nu)^2}
\]

\[
- \frac{\sin \lambda \nu \{lx - (m + i\beta \eta) y\} - \sinh \lambda (b - y) \sin lx - \sin \lambda \nu \{lx - (m + i\beta \eta) b\}}{\lambda^2 + (m + i\beta \nu)^2}
\]

(28)

in which the second member on the right is obtained from the first by changing the sign of \( m \), and prefixing a negative sign.

**Art. 5.** Preceding Result obtained more directly; corresponding value of \( \nu \) in two-dimensioned case.

The above result may, however, be obtained more directly from the fundamental hydrodynamic equations. These take the forms

\[
\frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} + \beta \nu = - \frac{1}{\rho} \frac{\partial p}{\partial x},
\]

\[
\frac{\partial v}{\partial t} + \beta \frac{\partial v}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial y},
\]

\[
\frac{\partial w}{\partial t} + \beta \frac{\partial w}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial z},
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

(29)

in which, as usual, \( p \) denotes pressure, and \( \rho \) density; and we require
solutions having wave-lengths \( \frac{2\pi}{l}, \frac{2\pi}{n} \) in the \( x \) and \( z \) directions. From these we obtain
\[
\left( \frac{d}{dt} + \beta y \frac{d}{dx} \right) \left( \frac{du}{dy} - \frac{dv}{dx} \right) + \beta \left( \frac{dv}{dx} + \frac{dw}{dy} \right) = 0, \quad \text{and}
\]
\[
\left( \frac{d}{dt} + \beta y \frac{d}{dx} \right) \left( \frac{dw}{dy} - \frac{dv}{dx} \right) + \beta \frac{dv}{dx} = 0
\]
and from these again, taken along with the equation of continuity, we obtain
\[
\frac{d}{dt} + \beta y \frac{d}{dx} \nabla^2 v = 0. \tag{31}
\]

The most general integral of this is
\[
\nabla^2 v = F(x - \beta \theta t, y, z), \tag{32}
\]
where \( F \) is an arbitrary function. With the initial value of \( v \), given by (24), this becomes
\[
\nabla^2 v = -(l^2 + m^2 + n^2) B \cos l(x - \beta \theta t) \sin m y \cos n z; \tag{33}
\]
and a particular value of \( v \) satisfying this is given by
\[
\frac{2v'}{(l^2 + m^2 + n^2) B \cos n z} = \frac{\sin \{lx + (m - l\beta t)y\}}{l^2 + (m - l\beta t)^2 + n^2} - \frac{\sin \{lx - (m + l\beta t)y\}}{l^2 + (m + l\beta t)^2 + n^2}. \tag{34}
\]
This, however, violates the conditions that \( v \) should vanish at the fixed planes \( y = 0, y = b \). We accordingly add to the value of \( v' \), as given by this equation, another, \( v'' \), satisfying the differential equation
\[
\nabla^2 v'' = 0, \tag{35}
\]
as well as the boundary conditions
\[
v'' = -v', \quad \text{when} \quad y = 0, y = b.
\]
This value of \( v'' \) obviously is given by
\[
\frac{2v'' \sinh \lambda b}{(l^2 + m^2 + n^2) B \cos n z} = \frac{-\sinh \lambda (b - y) \sin lx - \sinh \lambda y \sin \{lx + (m - l\beta t)b\}}{l^2 + (m - l\beta t)^2 + n^2} + \frac{\sinh \lambda (b - y) \sin lx + \sinh \lambda y \sin \{lx - (m + l\beta t)b\}}{l^2 + (m + l\beta t)^2 + n^2}
\]
\[
\tag{36}
\text{in which, as in (28),} \quad \lambda \text{ denotes } \sqrt{l^2 + n^2}. \quad \text{And the value } \quad v = v' + v'' \text{, obtained from (34) and (36), is identical with that given by (28).}
\]
If the solution of the three-dimensional problem is completed, the expressions for \( u, \omega \) involve a transcendental integral, and are somewhat longer than that found for \( v \). I accordingly return to the simpler case in which \( n \) is zero.

The initial values of \( u, v \) may now be written
\[
u_0 = \frac{-m B}{l} \sin lx \cos my \]
\[
v_0 = B \cos lx \sin my
\]
\[
[4^*]
\]
Proceedings of the Royal Irish Academy.

At time $t$ the value of $v$ is given by the two-dimensional form of (28), viz.:

$$
\frac{2v \sinh lb}{(l^2 + m^2)B} = \frac{\sinh lb \sin \{lx + (m - l\beta t)y\} - \sinh l(b - y) \sin lx \sinh ly \sin \{lx + (m - l\beta t)b\}}{l^2 + (m - l\beta t)^2} - \frac{\sinh lb \sin \{lx - (m + l\beta t)y\} - \sinh l(b - y) \sin lx \sinh ly \sin \{lx - (m + l\beta t)b\}}{l^2 + (m + l\beta t)^2};
$$

and that of $u$, obtained from $\frac{du}{dx} + \frac{dv}{dy} = 0$, is given by

$$
\frac{2lu \sinh lb}{(l^2 + m^2)B} = \frac{-(m - l\beta t) \sinh lb \sin \{lx + (m - l\beta t)y\} + l \cosh l(b - y) \cos lx - l \cosh ly \cos \{lx + (m - l\beta t)b\}}{l^2 + (m - l\beta t)^2} - \frac{(m + l\beta t) \sinh lb \sin \{lx - (m + l\beta t)y\} + l \cosh l(b - y) \cos lx - l \cosh ly \cos \{lx - (m + l\beta t)b\}}{l^2 + (m + l\beta t)^2};
$$

(38)

Art. 6. Reference to End-conditions; Example of Prescribed Conditions.

This, then, is the solution with the given initial conditions if the disturbance is to remain periodic in $x$ and of the assigned wave-length. It appears desirable, however, to allude to cases in which other and more definite conditions may be assigned at the ends of the stream. Suppose, for instance, with the same initial disturbance, it is made a condition that $u$ should vanish at two fixed planes $x = 0$, $x = a$, perpendicular to the direction of flow, in which case, of course, we must have $\sin lb = 0$ in order that this condition should be satisfied initially. We now add to the values $u$, $v$ given by (38) others $u_1$, $v_1$, which (i) satisfy equations (29), (ii) vanish everywhere initially in the region considered, (iii) make $v_1$ vanish at the planes $y = 0$, $y = b$, and (iv) make $u_1 = -u$ at the planes $x = 0$, $x = a$. These may be obtained as follows:—Denoting the value of $u$ as found from (38) at the planes $x = 0$, $x = a$, by $u_0(y, t)$, $u_a(y, t)$, respectively, expand these functions by Fourier's theorem in series of the forms

$$
u_a(y, t) = \sum_{r=1}^{\infty} B_r \cos \frac{r\pi y}{b},
$$

wherein the values of $r$ are positive integers and $A_r$, $B_r$, are functions of $t$. 

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which initially vanish as the original value of $u$ satisfies the end-conditions.

Find $C_r, D_r$, other functions of $t$, such that

$$C_r + D_r = A_r,$$

$$C_r e^{\pi a/b} + D_r e^{-\pi a/b} = B_r.$$  (40)

Take then

$$u_1 = -\sum \{ C_r e^{\pi x/b} + D_r e^{-\pi x/b} \cos r\pi y/b \},$$

$$v_1 = -\sum \{ C_r e^{\pi x/b} - D_r e^{-\pi x/b} \sin r\pi y/b \}.$$  (41)

these are the values to be added to $u, v$, as given by (38), in order to complete the solution.

In questions similar to that now under discussion, the use of infinite series, such as occur in (39), sometimes requires justification, especially in regard to differentiation. On such points reference may be made to Stokes' classical memoir.* In the present case, as the series in (39) converge, the form in which the exponential functions occur in the series in (41) shows that these latter series, as well as those formed of the differential coefficients of their successive terms with respect to $x$ or $y$, are uniformly convergent in the space considered; and in this space the differential coefficient of any order with respect to $x$ or $y$ of the sum of the series is evidently accordingly the sum of the differential coefficients of the separate terms; and thus the complete values of $u_1, v_1$, as well as the separate terms, satisfy the differential equations. The vanishing of the series for $u$, when $y = 0$ or $b$, or rather when $y$ is just inside these limits, does not follow from the mere fact that it is of the form $\sum a_r \sin r\pi y/b$, but is secured by the additional circumstance of its uniform convergence at these limits.

**Art. 7. Another Example of Prescribed End-conditions.**

As another example, if the prescribed end-conditions require $u$ to vanish at two planes $x - \beta by = 0$, $x - \beta by = a$, which move with the fluid, we replace the quantities $u_i, v_i$ just found, by others obtained in a similar manner from the values of $u$ of (38) at these planes instead of from its values at the fixed planes.

**Art. 8. The Solution found explains Instability.**

If the conditions to be satisfied at the ends of the stream are either those of (6) or those of (7) (and the same holds for many other conditions

---

which might be prescribed as alternatives), the results obtained afford a satisfactory explanation of the instability which is observed. Consider first the value of \( v \) as given in (28), without regard to end-conditions. The presence of the expressions

\[
\lambda^3 + (m - i\beta t)^3, \quad \lambda^3 + (m + i\beta t)^3,
\]

in the denominators shows, indeed, that \( v \) eventually diminishes indefinitely; but the occurrence of the former shows that before doing so the disturbance may increase, and increase in a very great ratio, or rather, as Professor Love has reminded me, that it may increase to such an extent that the equations (29), in which as usual only the first powers of \( u, v, w \) are retained, may cease to fairly represent the motion. If \( m \) is large compared with \( \lambda \), i.e., with \((P + q^3)\), then, as \( t \) approaches a value \( T \) given by \( m - i\beta T = 0 \), the second fraction in the right member of (28) becomes negligible compared with the first. At this particular instant of time the first term gives for \( v \) the approximate value

\[
v \approx \frac{\lambda^3 + m^3}{2\lambda^3} B \frac{\sinh \lambda b - \sinh \lambda (b - y) - \sinh \lambda y}{\sinh \lambda b} \sin \lambda t \cos \lambda z
\]

\[
= \frac{\lambda^3 + m^3}{2\lambda^3} B \left\{ 1 - \frac{\cosh \lambda (y - \frac{1}{2}b)}{\cosh \frac{1}{2}\lambda b} \right\} \sin \lambda t \cos \lambda z. \tag{42}
\]

The average value of this between the limits \( y = 0, \ y = b \) is

\[
\frac{\lambda^3 + m^3}{2\lambda^3} B \left\{ 1 - \frac{\tanh \frac{1}{2}\lambda b}{\frac{1}{2}\lambda b} \right\} \sin \lambda t \cos \lambda z.
\]

If \( \lambda/m \) and \( mb \) are each large, the ratio of this to the average initial value of \( v \) is great whatever be the value of \( \lambda b \). In the extreme case, in which \( \lambda b \) is very great, the ratio is approximately \( \pi m^2/4\lambda^3 \); in the other extreme case, in which \( \lambda b \) is very small, the ratio is approximately \( \pi m^2b/48 \).

In the two-dimensioned problem it may be seen that the average value of \( u \) does not increase in so great a ratio as that of \( v \). It should be noted, moreover, that at the critical time when \( m - i\beta t = 0 \) the most important part of \( u \) may be contributed by the second fraction in the right-hand member of (38), instead of by the first. In estimating the extent to which the disturbance as a whole is increased it must be borne in mind that, if \( m \) is large compared with \( t \), the original value of \( v \) is small compared with that of \( u \); so that the kinetic energy of the relative motion does not increase in so great a ratio as does \( v^2 \); it appears, in fact, that this energy increases in a ratio which is of order \( m^4/t^2 \) if \( lb \) is large, and of order \( m^2b^2 \) if \( lb \) is small. This follows most easily by using the stream-function. The velocity-components \( u, v \) are
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\[ \psi = \frac{d\phi}{dy}, \quad v = -\frac{d\psi}{dx}, \]  

where evidently \( \psi \) is given by the equation

\[ \begin{align*}
2\psi \sinh lb
&= \frac{\sinh lb \cos \{lx + (m - l\beta t)y\} - \sinh l(b - y) \cos lx - \sinh ly \cos \{lx + (m + l\beta t)b\}}{l^2 + (m - l\beta t)^2} \\
&\quad - \frac{\sinh lb \cos \{lx - (m + l\beta t)y\} - \sinh l(b - y) \cos lx - \sinh ly \cos \{lx - (m + l\beta t)b\}}{l^2 + (m + l\beta t)^2}.
\end{align*} \]  

(44)

If \( T \) be the average energy of the relative motion per unit length of pipe

\[ 4T = \int_0^l \int_0^b \left[ \left( \frac{d\psi}{dx} \right)^2 + \left( \frac{d\psi}{dy} \right)^2 \right] dx dy = \int \psi \frac{d\psi}{dn} dS - \int \nabla^2 \psi dS dy, \]  

(45)

the former integral being taken over the bounding surfaces. This integral is zero since \( \psi \) vanishes at the fixed planes \( y = 0, \quad y = b \), and since at the planes \( x = 0, \quad x = 2\pi/l \), the values of \( \psi \) are identical, and the values of \( \frac{d\psi}{dn} \) numerically equal, but of opposite signs. Thus we have only to deal with the final integral above wherein

\[ \nabla^2 \psi = B \frac{\beta^2 + m^2}{l} \sin l(x - \beta ty) \sin my. \]

If we retain only the first of the two fractions in the value of \( \psi \), as given by (44), we have, on integration with respect to \( x \),

\[ \frac{8\psi \sinh lb}{(l^2 + m^2)^2} \frac{T}{B} \int_0^l \int_0^b \left( \frac{d\psi}{dx} \right)^2 dx dy = \int \psi \frac{d\psi}{dn} dS - \int \nabla^2 \psi dS dy. \]  

(46)

At the critical time at which \( m - l\beta t \) is zero, we obtain on integration,

\[ T = Bb \frac{(l^2 + m^2)^2}{16\pi} \left( 1 - \frac{4m^2 \tanh \frac{1}{2l}}{l^2 + 4m^2} \right). \]

while originally \( T = Bb(l^2 + m^2)/8\pi \). Thus, \( m/l \) and \( mb \) being each large, the ratio of increase is great whatever be the value of \( lb \); its approximate values in the two extreme cases of \( lb \) great and \( lb \) small are respectively \( m^2/2\pi \) and \( m^3b/24 \).

These results are not substantially affected by conditions which may be prescribed at the ends of the stream, if the distance between them is large compared with \( \pi/l \). For example, if the end-conditions be those of Art. 6, the additional terms \( u_0, \quad v_0 \) of (41) are small compared with \( u \) of (38), except near the ends of the stream. This follows from the mode in which the exponential functions enter into (41).
It accordingly appears that, in this simple case, although the disturbance, if sufficiently small, must ultimately decrease indefinitely, yet, before doing so, it may be very much increased. By taking the wave-length at right angles to the direction of flow sufficiently small compared both with that in the direction of flow and with the distance between the fixed boundaries, the ratio of increase may be made as great as we like, provided, that is, the approximate equations (29) continue to fairly represent the motion. Unless, then, the limits within which these equations do hold increase indefinitely as \( m/l \) and \( m \phi \) increase, these limits may be exceeded. As Professor Love has pointed out to me, the possibility of passing these limits does not afford a thoroughly satisfactory proof of instability, but merely shows that the disturbance will increase until the equations cease to represent the motion. A rigorous proof that a state of motion or of equilibrium is unstable, is thus, in many cases, a matter of excessive difficulty; but a result such as obtained here may, I think, be regarded as strong \( \text{a priori} \) evidence of instability and as a satisfactory \( \text{a posteriori} \) explanation of an actually observed instability.

**Art. 9. Practical Instability of Motion is consistent with Stability for Principal Modes of Disturbance.**

At first sight, it may appear that the possibility of an arbitrary disturbance being unstable is inconsistent with the stability of the fundamental oscillations into which it can be resolved; but, on consideration, it may be seen that there is no inconsistency, and that in reality, when a system possesses an infinite number of coordinates, the stability of its fundamental modes of oscillation, whether about a state of steady motion or one of equilibrium, affords no proof that it is stable for an arbitrary disturbance. Fourier’s analysis proves, in fact, that an infinite series of the type

\[
\sum (C_r \cos \omega_r t + S_r \sin \omega_r t)
\]

may, at times, have values very great compared with its initial one, and may even become infinite. If the question is that of the stability of a given state of equilibrium, and the system possesses a potential energy-function, it is, in reality, settled by the form of this function. By the well-known argument, the sum of the kinetic and potential energies is constant in any motion; and, accordingly, if the latter is a minimum in the position of equilibrium, the system can never deviate so far from this position that the potential energy should exceed the sum of the potential and kinetic energies of the initial disturbance. If we endeavour to answer the question by ascertaining the nature of the roots of the equation which gives the periods of the free
disturbances, the reality of all the values of $\omega$ is a satisfactory proof of
stability, only for the reason that it shows that, if there be a potential-energy
function it is essentially positive in any displacement (if taken as zero in
the equilibrium position); for the problem of finding the free periods is
analytically identical with that of transforming the coordinates, so that the
kinetic energy can be expressed as a linear function of the squares of the
velocities, and at the same time the potential energy as a linear function of
the squares of the coordinates, the terms involving products being thus made
to disappear; and if all the periods are real, each coefficient in the potential
energy-function is positive. Whether the potential energy is, or is not, a
minimum in the state of equilibrium can, of course, generally be decided
much more easily directly than by investigating the free periods.

If we consider even a system having only a finite number of coordinates,
and which is slightly displaced from equilibrium, the argument for universal
stability which is derivable from the stability of the fundamental modes may
be very much weakened (i.e., the limits of stability may be very much
narrowed) by the non-existence of a potential-energy function. Take, for
example, two particles of equal mass oscillating in a straight line, and
subject to forces such that the most general small motion is given by the
equations

$$x = A \cos (pt + a) + B \cos (qt + \beta),$$
$$y = A \cos (pt + a) + Bk \cos (qt + \beta),$$

wherein $A, B, a, \beta$ are arbitrary, but $k$ is a definite constant, nearly equal to
unity. Suppose that in the position of equilibrium a velocity is imparted to
the second particle only. The resulting motion is given by

$$x = A \left( \sin pt - \frac{p}{q} \sin qt \right),$$
$$y = A \left( \sin pt - kq \frac{p}{q} \sin qt \right);$$

and we see that if $p, q$ are such that we can have simultaneously $\cos pt = + 1$
$\cos qt = - 1$, the maximum kinetic energy exceeds the initial in the ratio
$(5 + 2k + k^2)/(1 - k)^2$, which may be exceedingly great. If, however, the same
arbitrary constants $A, B, a, \beta$ occur in the equations expressing the small
motions of a similar system having a potential-energy-function, $k$ must have
the value $- 1$; and in consequence, if the system be started subject to the
same initial conditions, the kinetic energy can never exceed its initial value.

When the question is of the stability of a given state of motion, if the
state is one for which the sum of the kinetic and potential energies is a
minimum or maximum, then, whether steady or not, it is stable; for if the

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system is started in a slightly different state, its subsequent motion is confined to those slightly different states for what the total energy differs from the maximum or minimum value by the same amount as at starting. This general theorem, like the energy-test of the stability of equilibrium, applies to cases in which the number of coordinates is infinite; but in steady motion, although it thus appears that the periods of the fundamental free disturbances are real if the total energy is a maximum or a minimum, yet, in contradistinction to equilibrium, the converse is not true; the free periods may be real and yet the energy not a maximum or minimum. As far as I am aware, no theorem imposing any limitation on the amount of deviation from the steady state which is possible when nothing more is known than that the free periods are real, has been established in such a form as to hold when the number of coordinates is infinite; and accordingly I think it has not been established that in such a case reality of the free periods constitutes a sufficient condition of stability, and this whether there is, or is not, a potential energy-function.

Even when the number of coordinates is small—as small as two—the system may be such that a large deviation from the steady state may result from a small initial disturbance, although the periods are real and very unequal. (It is, of course, known that this may happen if two periods are nearly equal.)

Suppose, for instance, a system in which the most general deviation from the steady motion is expressed by the equations

\[ x = a \cos(pt + a) + b \cos(qt + \beta), \]
\[ y = a \sin(pt + a) + k\delta q^{-1} \sin(qt + \beta), \]

wherein \( x, y \) are coordinates which vanish in the steady motion, and \( a, b, a, \beta \) are arbitrary constants, but \( k \) is a definite constant, nearly equal to unity. Suppose the particular solution taken is

\[ x = a (\cos pt - \cos qt), \]
\[ y = a \left( \sin pt - \frac{b\delta}{q} \sin qt \right), \]

giving

\[ \dot{x} = a (-p \sin pt + q \sin qt), \]
\[ \dot{y} = ap (\cos pt - k \cos qt). \]

The system starts in a position which occurs in the steady motion and with a disturbed velocity in the \( y \) coordinate alone; and we see that if \( p \) and \( q \) are such that we can have simultaneously \( \cos pt = +1, \cos qt = -1 \), the disturbance in the velocity in this direction exceeds its initial value in the ratio \( (1 + k)/(1 - k) \), which may be exceedingly great.

It is easy to formulate, and in a variety of ways, kinetic and potential energy-functions which lead to the above solutions,
A concrete physical example may be given. Routh discusses the following problem* :— "A body has a point 0 which is in one of the principal axes at the centre of gravity G fixed in space. The body is in steady motion rotating with angular velocity \( \omega \) about OG, which is vertical. Find the conditions that the motion may be stable."

When the deviations are made to vary as \( e^{it} \), the resulting equations are

\[
\begin{align*}
((A - C) n^2 + Mg/h + Bp^2) \xi + (A + B - C) imp = 0, \\
-(A + B - C) imp \xi + ((B - C) n^2 + Mg/h + Ap^2) \eta = 0,
\end{align*}
\]

\( \xi, \eta \), being the direction-cosines of the vertical referred to \( OA, OB \), and \( h \) the height of \( G \) above \( O \). Routh investigates the condition when \( A = B \), a case which could not be made to suit the present requirements. We may, however, simplify what follows by supposing \( C = A \), when the equations become

\[
(Mgh + Bp^2) \xi + B imp \eta = 0, \\
-B imp \xi + ((B - A) n^2 + Mg/h + Ap^2) \eta = 0.
\]

Evidently what is required for the possibility of a solution of the type cited is that, in the two fundamental oscillations, the two values of the quotient of \( \xi \) by \( \eta \) should be nearly equal, and yet the two values of \( p \) not nearly equal. This requirement is satisfied if the two values of \( Bp^2 \) are small compared with \( Mg/h \), and yet not nearly equal. The equation determining \( p \) is

\[
(Bp^2 + Mg/h) [Ap^2 + Mg/h - (A - B) n^2] - B^2n^2p^2 = 0.
\]

Evidently it is necessary for stability that \( Mg/h - (A - B) n^2 \) should be positive; and we will suppose \( A > B \). Considering the equation

\[
(p^2 + \alpha)(p^2 + \beta) - \gamma p^2 = 0,
\]

where \( \alpha, \beta, \gamma \) are positive, we see that if, for example,

\[
\sqrt{\beta} = \epsilon \sqrt{\alpha}, \\
\sqrt{\gamma} = (1 + 2\epsilon) \sqrt{\alpha},
\]

\( \epsilon \) being small, the two values of \( p^2 \), namely,

\[
\frac{\alpha}{2} \left\{ 4\epsilon + 3\epsilon^2 \pm \sqrt{(4\epsilon + 3\epsilon^2)^2 - 4\epsilon^2} \right\},
\]

are real, positive, very unequal, and small compared with \( \alpha \). Applying the above conditions to the case in point, they are equivalent to

\[
\frac{Mgh - (A - B)n^2}{Mgh} \cdot \frac{B}{A} = \epsilon^2,
\]

\[
\frac{Bp^2}{Amgh} = (1 + 2\epsilon)^2,
\]

which lead, by elimination, to the following relation between \( A \) and \( B \):

\[
\{Bp^2 - (A - B) A (1 + 2\epsilon)^2\} = \epsilon^2 AB.
\]

This gives a value for \( B/A \) which is nearly equal to \((\sqrt{5} - 1)/2\).

---

* "Stability of a given State of Motion," p. 64.
It appears then that the body may be such, and so moving, that, in spite of the reality of the free periods, a small initial disturbance of the steady motion may lead at some time to a large one, that is as far as can be ascertained by equations which take account only of the first powers of small quantities.

In this example, as well as in the preceding one relative to a state of equilibrium, the value of \( k \) cannot, of course, ever be equal to unity exactly; in this limiting case the values of \( p, q \) become equal, and the solution of the equations of motion assumes a different form;* so that another mode of contrasting the case of equilibrium when there is an energy-function with those of equilibrium when there is no energy-function and of steady motion is to say that, in the former case, equality of periods cannot be destructive of stability, but in the others it may; and also that in the others the evil effects of what may be regarded as in reality an approach to equality of periods cannot be estimated by regard to the ratio of the periods alone. And in this connexion it may be borne in mind that, in the liquid system under discussion, we have an extreme case of the equality of free periods, as their values range continuously from one limit to another.

**ART. 10. The bearing of the Non-Vanishing of the Integrated Product of Velocities in two Principal Modes.**

The fact that for some types of disturbance the steady motion may be practically unstable in spite of the stability of the fundamental modes may be seen to be connected with another fact noted above (p. 24), namely, that if \( v_1 \cos \lambda (x - U_1 t), \ v_2 \cos \lambda (x - U_2 t) \) denote the values of \( v \) in two fundamental modes having the same wave-length in the \( x \)-direction, we do not have, as in the case of a system possessing a potential-energy function and oscillating about a position of equilibrium, the relation

\[
\int_0^b v_1 v_2 dy = 0.
\]

If this relation did hold, it is easily seen that the total kinetic energy due to the velocity in the \( y \)-direction would be independent of the time, whereas in the actual case there occur terms of the type

\[
\frac{\cos \lambda}{\sin \lambda} (U_1 - U_2) t \int_0^b v_1 v_2 dy,
\]

whose value at the time \( t \) may be large compared with their initial value.

* And in this limiting form the expressions for the coordinates contain terms proportional to \( t \cos pt, t \sin pt; \) equality of periods introduces no such terms into the solution of the equations of the small motions of a system displaced from equilibrium if \( p \) possesses an energy-function.
Similarly, in the two-dimensional problem, the kinetic energy due to x-component of relative velocity would be independent of the time, provided
\[ \int_0^b u_x u_x dy = 0, \]
or
\[ \int_0^b \frac{dv_1}{dy} \cdot \frac{dv_2}{dy} dy = 0, \]
which relation does not actually hold.

And, in the two-dimensional problem, the total kinetic energy of the relative motion involving both x and y components of velocity would be independent of the time, provided for every two fundamental modes,
\[ \int_{2\pi/\lambda} \int_0^b \left( \frac{d\phi_2}{dx} \cdot \frac{d\phi_2}{dy} + \frac{d\phi_1}{dy} \cdot \frac{d\phi_2}{dy} \right) dy = 0. \]
This integral is seen to reduce to
\[ \int_{2\pi/\lambda} \int_0^b \frac{d\phi_2}{dy} \left| \frac{d\phi_2}{dy} \right| dx, \]
where \( |d\phi_2/dy| \) denotes the discontinuity in the value of \( d\phi_2/dy \) at the plane where slipping occurs in the corresponding fundamental disturbance, and accordingly does not vanish as a rule.

**Art. 11. Energy of Actual Motion increases; Work is done by End-Pressures.**

It has been shown that the kinetic energy of the relative motion of a disturbance may increase, and the same is true for the energy of the actual motion. When the disturbance is periodic in x, the energy of the total motion is, in fact, equal to that of the steady motion, together with that of the relative motion. The difference in fact is \( \int \int \beta y u dx dy \); and in estimating this correctly to the second order of small quantities, terms of the second order in \( u \) must be taken account of. To whatever order, however, the approximation is made, this integral is zero if taken through a range \( 2\pi/l \) in \( x \).*

Now, a change in the energy within a given space may be caused either by the energies of the entering fluid and of that which is flowing out being different, or by the rate at which the boundary-pressures do work on the contained fluid being other than zero. In a length \( 2\pi/l \) the first cause is ineffective, as the velocities at the two ends are identical in value; and as the

*As far as \( x \) is involved, some of the second order terms in \( u \) are constant, and others have a period \( \pi/l \); \( \int \int \beta y u dx dy \) will vanish only if the former terms are annulled, which may of course be done by suitable end-conditions, as any function of \( y \) can be added to \( u \).
pressure also is to the first order of small quantities periodic, it may appear paradoxical that the second cause should have any effect either. In computing, however, to the second order of small quantities the rate at which the pressures do work on the fluid, terms of the second order must be retained in the pressure. And to this order the pressure is not periodic in \(x\), as is shown by the equation

\[
-\frac{1}{\rho} \frac{dp}{dx} = \frac{du}{dt} + (\beta y + v) \frac{du}{dx} + v \frac{du}{dy},
\]

(47)

for the product \(v du/dy\) involves \(\sin^2 lx\) and \(\cos^2 lx\).

**Art. 12. The Motion is Stable, if Initial Disturbance be sufficiently small.**

It is evident, then, from what precedes, that Lord Rayleigh's analysis is sufficient to include the most general disturbance. And as the former of equations (30) is now equivalent to

\[\nabla^2 \psi = f(x - \beta ty, y),\]

and leads to an expression for \(\psi\) in terms of integrals which are obviously finite, unless the end-conditions are extraordinary, it appears that, as long as equations (29) represent the motion, a disturbance cannot increase indefinitely, and accordingly that the motion is stable for the most general disturbance, if sufficiently small initially.

Equation (32) shows also that, in the case of a disturbance in three dimensions, the same is true at least as far as \(v\) is concerned; and it seems reasonable to infer stability for a sufficiently small disturbance of this type also.

Indeed, if the disturbance is of definite wave-lengths in the \(x\) and \(z\) directions, but is of an arbitrary character in so far as it depends on \(y\), it may be seen that, if sufficiently small initially, the \(y\) velocity-component eventually diminishes indefinitely as \(t^2\), and, in the two-dimensioned case at least, the \(x\) component of relative velocity as \(t^2\).

If the disturbance has initially

\[
v_0 = f(y) \cos lx \cos nz,
\]

(48)

then at time \(t\) we have, (see equation (23)):

\[
2v/\cos zn = \sinh \lambda (b - y) \int_y^b \sinh \lambda \eta \left\{ \lambda f(\eta) - f'f''(\eta) \right\} \cos l(x - \beta \eta t) d\eta
\]

\[
+ \sinh \lambda y \int_y^b \sinh \lambda (b - \eta) \left\{ \lambda f(\eta) - f''(\eta) \right\} \cos l(x - \beta \eta t) d\eta
\]

\[
+ \text{terms derivable by changing } x - \beta \eta t \text{ into } x + \beta \eta t,
\]

where \(\lambda^2 = \beta^2 + n^2\).

(49)
Fixing attention on the first and second terms alone on the right, and writing them in the form
\[ \sinh \lambda (b - y) \int_{\eta}^{b} U \cos l(x - \beta \eta t) \, d\eta + \sinh \lambda y \int_{y}^{b} V \cos l(x - \beta \eta t) \, d\eta, \]
on integration by parts, since the terms at the limits cancel, this becomes
\[ (l\beta t)^{-1} \left[ \sinh \lambda (b - y) \int_{\eta}^{b} \frac{dU}{d\eta} \sin l(x - \beta \eta t) \, d\eta + \sinh \lambda y \int_{y}^{b} \frac{dV}{d\eta} \sin l(x - \beta \eta t) \, d\eta \right]. \]
(51)

On integration again by parts, we obtain terms at the limits varying as \( t^{-1} \), which do not cancel, and also integrals which, when \( t \) is sufficiently great, may be proved to be negligible in comparison with those terms.

The third and fourth terms of \( v \) may be treated similarly; and the result stated as to the ultimate form of the value of \( v \) thus follows.

And, in the two-dimensioned case, the corresponding value of \( u \) at time \( t \) is evidently given by
\[ 2u = \cosh \lambda (b - y) \int_{\eta}^{b} U \sin l(x - \beta \eta t) \, d\eta - \cosh \lambda y \int_{y}^{b} V \sin l(x - \beta \eta t) \, d\eta \]
+ two other terms.
(52)

On integration by parts in the same manner, we obtain terms at the limits which do not cancel, and vary ultimately as \( t^{-1} \), and integrals which, when \( t \) is large enough, may be neglected in comparison with those terms.

**Art. 13. Case of Several Layers of Constant, but Different, Vorticities.**

I proceed to allude briefly to the more general case, in which the stream is composed of a number of layers, each having constant, but different, vorticities, and there being no slipping at the surfaces of transition. Equation (31) holds for each layer; and its first integral throughout may be written
\[ \nabla^2 v = F(x - Ut, y, z), \]
(53)

\( U \) being in any layer of the form \( \beta y + c \), with different values of \( \beta, c \) in each layer. If we take a two-dimensional disturbance, in which initially
\[ v_0 = 2 \cos lx \sin my/(l^2 + m^2) \]
(54)

we have, at time \( t \),
\[ \nabla^2 v = - \sin l(x - Ut + my) + \sin l(x - Ut - my). \]
(55)

For brevity, consider only the first term; this, of course, corresponds to a wave which might occur alone. This leads to, in any layer,
\[ v = \frac{\sin l(x - Ut + my)}{l^2 + (m - l\beta)^2} + v', \]
(56)
where \( \nabla^2 v' = 0 \), and the values of \( v' \) are such that \( v \) satisfies (10), (11), that \( v \) vanishes at the fixed bounding planes, and that \( v' \) initially vanishes everywhere. Evidently in each layer \( v' \) is of the form
\[
v' = \sinh ly \left[ F(t) \cos lx + f(t) \sin lx \right] + \cosh ly \left[ \phi(t) \cos lx + \psi(t) \sin lx \right],
\]
where the functions of \( t \) have all to be determined.

Equations (10), (11) give at each surface of separation four relations among the functions and their differential coefficients with respect to time, which correspond to the regions meeting there. The vanishing of \( v \) at the fixed boundaries gives four other equations. There are thus obtained as many equations as there are functions of \( t \). (These equations differ from those obtained in Lord Rayleigh's investigation of the fundamental oscillations by having, when all the unknown functions are brought to the left-hand side, as their right-hand members given functions of the time instead of zero.) The value of \( v' \), and therefore that of \( v \), is evidently determinate; and the solution is unique; for if \( v' + v'' \) be substituted for \( v' \), it appears that \( v'' \) must satisfy (10) and (11), must vanish at the boundaries, and be initially zero everywhere, that is, it must represent a free oscillation which is initially zero, and must therefore be zero always.

Thus, in this case also, the analysis which has been given suffices to include the most general disturbance possible. The complete determination of \( v \), even for an initial disturbance of the simple type discussed in the case of uniform shearing, involves transcendental integrals. The expression for \( u \) could be easily written down when that for \( v \) is obtained.

Now, the form of the expressions for \( u, v \) shows that in this case also the disturbance may increase very much. The first term in \( v \) will as before increase very much if \( m/l \) is large; and the hyperbolic functions in \( v' \) show that if \( l \) times the thickness of the layer is large, \( v' \) could neutralize this first term in the neighbourhood of two planes only. It is not so clearly evident, however, that, as in the simpler case, the disturbance may increase greatly, even if \( l \) times the thickness of the layer is small, provided \( m \) times it is large. It seems, however, reasonable to suppose that, if the initial wavelength measured at right angles to the layer is small compared with the thickness of the layer, the conditions of stability can depend little on the conditions at the boundaries of the layer, and that therefore, in the cases in which, as we have seen, the motion may be unstable when those boundaries behave as fixed walls, it would also be unstable when the conditions to be
satisfied at them are those which prevail when the stream is disturbed through its entire thickness. If this argument is legitimate, even the brief discussion of the forms of \( u, v \) which has been given might be dispensed with.

**Art. 14. The Case of Continuous, but Varying, Vorticity.**

The explanation just given of the possibility of instability in the case of a finite number of layers cannot, at least *prima facie*, by making the number of layers infinite, be extended to cover the case of continuously varying vorticity. For, as presented above, it requires at least that the original wave-length at right angles to the stream should be small compared with the thickness of some layer.

In this general case, confining ourselves to two dimensions and using the stream-function \( \psi \), we readily obtain instead of (31) the more general equation

\[
\left( \frac{d}{dt} + U \frac{d}{dx} \right) \nabla^2 \psi - \frac{d\psi}{dx} \frac{d^2 U}{dy^2} = 0. \tag{57}
\]

This equation is intractable, and, as has been seen, the consideration of disturbances alone which vary as \( e^{int} \) is not sufficient; but some light may be thrown on the question under discussion by considering a certain type of approximate solution. The approximate solution of a differential equation when an accurate one is not feasible is, however, a question of considerable delicacy. Let us endeavour to see under what conditions this equation would be satisfied by the approximate value

\[
\psi \approx \frac{\cos \{ l(x - Ut) + my \} }{l^2 + (m - i\delta U/\partial y)^2}, \tag{58}
\]

which of course implies regarding \( dU/\partial y \) as a constant. One might be disposed to state that the necessary conditions are that the terms neglected should be small compared with those which are retained either in \( d/\partial t \cdot \nabla^2 \psi \), or in \( U \frac{d}{dx} \cdot \nabla^2 \psi \), i.e. with \( lU \). Evidently, however, the addition or subtraction of a constant to or from \( U \) should leave the problem unaltered (or at most require only some modification of the end-conditions).* In any equation indeed, algebraic or differential, the division into terms is to some extent a matter of convenience; and if we strike out a term, it is not quite

* It seems evident that some consideration of end-conditions in all these problems is desirable, if we reflect that by ignoring them we might reduce the question of the stability of a stream of uniform velocity to that of a liquid at rest. These questions are, it seems obvious, practically different.

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clear what ratio we are neglecting; it thus may be difficult to say how far one approximation to a solution is good without obtaining a closer one or the accurate one. It is, I think, reasonable to require the terms neglected to be small compared with \( l(u_1 - u_2) \), where \( u_1, u_2 \) are the greatest and least values of \( U \), instead of with \( lU \). The term \( \frac{d\psi}{dx} \cdot \frac{d^2U}{dy^2} \), which is neglected, is of order \( \frac{d^2U}{dy^2} \mid I \) at the time when \( m - ltdU/dy \) is zero: neglecting this term is thus equivalent to neglecting \( \frac{d^3U}{dy^3} \), which would usually be of order \( 1/lb \). Again, as to the terms neglected in \( (d/dt + Ud/xd)\nabla^2\psi \), with the approximate value of \( \psi \), which has been taken,

\[
\frac{d\psi}{dx} = -\frac{l \sin \{l(x - Ut) + my\}}{\Psi + (m - ltdU/dy)^2},
\]

\[
\frac{d\psi}{dy} = -\frac{(m - ltdU/dy) \sin \{l(x - Ut) + my\}}{\Psi + (m - ltdU/dy)^2} + \frac{2ltdU/dy^2(m - ltdU/dy) \cos \{l(x - Ut) + my\}}{[\Psi + (m - ltdU/dy)^2]^2}.
\]

Apparently we may fairly neglect \( d^3U/dy^3 \) in \( d\psi/dy \) and in the succeeding differentiations, if the second term in \( d\psi/dy \) is small compared with the first, i.e. if \( ltd^2U/dy^2 \) is small compared with \( \Psi + (m - ltdU/dy)^2 \), and if we wish this to be so up to the time when \( m - ltdU/dy \) is zero, we neglect \( \frac{md^3U/dy^3}{\Psi dU/dy} \), which is usually of order \( m/lb \). I think then, that under these conditions, the \( \psi \) of (58) may be taken as an approximate solution of the equation (57). It, however, violates the conditions of vanishing at the bounding planes \( y = 0, y = b \). As stated in the previous Article, there is reason to think that this objection might be ignored if \( mb \) be large.

Now, although this value of \( \psi \) eventually decreases indefinitely, yet if \( m/l \) is large, before decreasing it increases very much, approximately in the ratio \( m^2/\Psi \), the maximum value at any place being obtained at a time when \( m - ltdU/dy \) is zero. There is thus, I think, evidence of possible instability in the most general case and whether \( d^3U/dy^3 \) be one signed or not; I do not of course regard this discussion as containing a satisfactory proof. The case for instability is further weakened by the circumstance that the critical time at which \( \psi \) is greatest now depends on \( y \).
CHAPTER II.

THE CASE OF FLOW THROUGH A PIPE WHOSE SECTION IS A CIRCLE OR TWO CONCENTRIC CIRCLES.

Art. 15. Lord Rayleigh's Investigation.

Lord Rayleigh has discussed also the question of the stability of steady flow through a pipe of circular section, or an annular pipe whose section is two concentric circles, and has concluded that when the undisturbed motion is that appropriate to a viscous fluid no disturbance of the steady motion is exponentially unstable, provided viscosity be altogether ignored. It seems desirable to quote the substance of his discussion at least for a disturbance symmetrical about the axis. Referring the motion to cylindrical coordinates \( z, r, \theta \), parallel to which the component velocities are \( w, u, 0 \), we have

\[
\frac{Du}{Dt} = \frac{dQ}{dr}, \quad \frac{Dw}{Dt} = \frac{dQ}{dz}, \quad D|Dt| = \frac{d}{dt} + ud\frac{dr}{dz} + wd\frac{dz}{dr},
\]

where \( -Q = V + p/\rho \), and \( V \) is the potential of the impressed forces. In applying these general equations to the present problem of small disturbances from a steady motion represented by \( u = 0, w = W \), where \( W \) is a function of \( r \) only, the complete motion is regarded as expressed by \( u, W + w \), and the squares of the small quantities \( u, w \) are neglected.

Thus:

\[
\frac{du}{dt} + W\frac{du}{dz} = \frac{dQ}{dr}, \quad (1)
\]

\[
\frac{dw}{dt} + ud\frac{W}{dr} + W\frac{dw}{dz} = \frac{dQ}{dz}, \quad (2)
\]

which, with the equation of continuity,

\[
\frac{d}{dr}(ru) + rd\frac{w}{dr} = 0, \quad (3)
\]

determine the motion.

The next step is to introduce the supposition that, as functions of \( t, z \), the variables \( u, w, Q \) are proportional to \( e^{(nt+kz)} \).

This gives

\[
i(n + kW)u = dQ/dr, \quad (4)
\]

\[
u(W) + i(n + kW)w = ikQ, \quad (5)
\]

\[
d(ru)/dr + ikrw = 0. \quad (6)
\]

Eliminating \( w, Q \), there is obtained the equation

\[
(n + kW)\left\{\frac{du}{d\rho} + \frac{1}{r} \frac{du}{dr} - u - k^2 w\right\} - ku\left\{\frac{d^2 W}{dr^2} - \frac{1}{r} \frac{dW}{dr}\right\} = 0. \quad (7)
\]

If the undisturbed motion be that of a viscous fluid, \( W \) is of the form

\[\text{[6*]}\]
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$A + Br^s$, and the second part of the left-hand member of (7) disappears. There can then be admitted no values of $n$, except such as make $n + kW = 0$ for some value of $r$ included within the tube. For the equation

$$\frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} - kW = 0; \tag{8}$$

being that of the Bessel's function of first order with a purely imaginary argument, $[I_i(kr)]$, admits of no solution consistent with the conditions [requisite when the section is a circle], that $u = 0$ when $r$ vanishes, and also when $r$ has the finite value appropriate to the wall of the tube [or consistent with the conditions, which must be satisfied in an annular tube, that $u = 0$ for two real finite values of $r$]. But any value assumed by $-kW$ is an admissible solution for $n$. At the place where $n + kW = 0$, (8) need not be satisfied; and under this exemption the required solution may be obtained consistently with the boundary conditions.* It is included in the above statement that no admissible value of $n$ can include an imaginary part.

Lord Rayleigh then proceeds to consider disturbances which are unsymmetrical. Taking $u, v, w, Q$ to be proportional to $e^{i(n+kr+\theta)}$, the equation which replaces (7) is highly intractable; he shows, however, that no complex value of $n$ is admissible.† This result is also established when $W$ is any function of $r$ whatever, provided

$$\frac{d^2W}{dr^2} + \frac{1}{r} \frac{dW}{dr} - kW = 0,$$

is of one sign throughout the region.

Art. 16. An Arbitrary Symmetrical Disturbance resolved into a Series of Lord Rayleigh's Type, when Law of Flow is that of Viscous Liquid in Complete Pipe.

It is seen from the above that there is only one law of steady motion which can be fairly said to lend itself to an analytical investigation, and this only when the disturbance is symmetrical; this law, however, is at the same time that which is of the greatest interest physically, as being that which governs the steady flow of viscous liquid through a circular pipe, viz.:—

$W = A + Br^s$. Taking, then, this case, I proceed to show that Lord Rayleigh's analysis suffices for the discussion of the most general disturbance, which is

* As in the corresponding case of plane strata (Chap. I., Art. 1), Lord Rayleigh obviously implies that in the regions separated by the surface for which $n + kW$ vanishes, different solutions of (8) are to be taken and fitted together so as to make $u$ continuous. This, of course, necessitates slippage at the dividing surface.
† At this point in Lord Rayleigh's investigation there is a slight error which does not affect his conclusion. He regards (Collected Papers, iii., p. 586, l. 3) a certain function of $r$ as a fixed number.
symmetrical about the axis, and to examine the propagation of one initially of type analytically simple. In the more general case of an annular tube, whose outer and inner radii are \( a, b \), the problem in expansions may evidently be reduced to the following:—

Given \( f(r) \) an arbitrary function of \( r \), find a function \( \phi \) such that for values of \( r \) between \( b \) and \( a \)

\[
f(r) = \int_b^a \phi(\rho) F_\rho(r) \, d\rho,
\]

where \( F_\rho(r) \) is a given function of \( r \), defined thus: when \( r \) lies between \( b \) and \( \rho \)

\[
F_\rho(r) = A \{ I_1(kr) K_1(kb) - I_1(kb) K_1(kr) \},
\]

and when \( r \) lies between \( \rho \) and \( a \)

\[
F_\rho(r) = B \{ I_1(\rho r) K_1(\rho a) - I_1(\rho a) K_1(\rho r) \},
\]

\( K_1(\rho r) \) denoting the solution of (8) which vanishes when \( r \) is infinite, \( k \) being a given constant, and the values of \( A, B \) being so connected that the two forms of \( F_\rho(r) \) have identical values at their common limit, \( r = \rho \).

As there is an arbitrary multiplier in \( K_1(\rho r) \), and as any convenient values of \( A, B \) may be chosen, we will take \( K_1(\rho r) \) to be that solution of (8) which, for a large real positive value of \( kr \), approximates to \( \frac{\pi}{4} (2kr)^{-\delta} e^{-kr} \); and we will take

\[
A = I_1(\rho p) K_1(\rho a) - I_1(\rho a) K_1(\rho p),
\]

\[
B = I_1(\rho p) K_1(\rho b) - I_1(\rho b) K_1(\rho p).
\]

The form of \( \phi(\rho) \) may be discovered by a procedure similar to that of Art. 4, and found to be given by the equation

\[
- \phi(\rho) \{ I_1(\rho a) K_1(\rho b) - I_1(\rho b) K_1(\rho a) \} = \rho \left( k^2 + \frac{1}{\rho^2} \right) f(\rho) - f'(\rho) - \rho f''(\rho).
\]

The equation expressing the expansion, if any such be possible, is thus

\[
- \{ I_1(\rho a) K_1(\rho b) - I_1(\rho b) K_1(\rho a) \} f(r) = \int_b^a \left\{ \rho \left( k^2 + \frac{1}{\rho^2} \right) f(\rho) - f'(\rho) - \rho f''(\rho) \right\} F_\rho(r) \, d\rho,
\]

and it may be easily verified that this is true, provided that \( f(r) \) and \( f'(r) \) are finite, continuous, and differentiable throughout the region, and that \( f(\rho) \) vanishes at both the boundaries \( \rho = a, \rho = b \). The most general symmetrical disturbance can thus be analysed into elementary ones of Lord Rayleigh's type.
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For a disturbance varying as $e^{ikr}$, each element of the integral in (15) is then to be multiplied by $e^{ikr}$. And the result obtained is that, if initially the disturbance is given by $u = e^{ikr}f(r)$, then, at time $t$, it is given by

$$- \{I_t(ko)K_t(kk) - I_t(kk)K_t(ko)\} u = e^{ikr} \left( I_t(kr)K_t(kk) - I_t(kk)K_t(kr) \right) \int \left( (\rho^2 + \rho^2) f(\rho) - f'(\rho) - \rho f''(\rho) \right) \{I_t(kp)K_t(kb) - I_t(kb)K_t(kp)\} e^{ikr}d\rho,$$

another term obtainable from this by interchanging $a, b$, (16) the argument in $W$ being $\rho$.

Art. 17. The preceding Result obtained more directly.

The result to which this analysis would lead may, however, as in the plane case, be obtained more directly from the fundamental equations. Eliminating $Q$ from (1), (2), we obtain

$$\left( \frac{d}{dt} + W \frac{d}{dz} \right) \left( \frac{du}{dz} - \frac{du}{dr} \right) + \left( \frac{du}{dz} + \frac{du}{dr} \right) \frac{dW}{dr} + u \frac{d^2W}{dr^2} = 0; \quad (17)$$

this equation is, as far as terms of the first order of small quantities, the equivalent of

$$\left( \frac{d}{dt} + (W + u) \frac{d}{dz} \right) \left( \frac{du}{dz} - \frac{du}{dr} \right) + \left( \frac{du}{dz} + \frac{du}{dr} \right) \left( \frac{d(W + u)}{dr} - \frac{du}{dz} \right) = 0, \quad (18)$$

which expresses the constancy of the vortex strength.† By using (3), (17) becomes

$$\left( \frac{d}{dt} + W \frac{d}{dz} \right) \left( \frac{du}{dz} - \frac{du}{dr} \right) + u \left( \frac{d^2W}{dr^2} - \frac{1}{r} \frac{dW}{dr} \right) = 0; \quad (19)$$

and, again using (3), we obtain

$$\left( \frac{d}{dt} + W \frac{d}{dz} \right) \left( \frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} + \frac{d^2u}{dz^2} \right) - \frac{du}{dz} \left( \frac{d^2W}{dr^2} - \frac{1}{r^2} \frac{dW}{dr} \right) = 0, \quad (20)$$

an equation of which (7) is a particular instance. For the form of $W$ with which we are dealing, the second part of the left-hand member vanishes; and we obtain as an integral

$$\frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} + \frac{d^2u}{dz^2} = F(z - Wt, r), \quad (21)$$

where $F$ may be any function, but is determined from the initial values of $u$.

If we now introduce the supposition that $u$ as a function of $z$ is proportional to $e^{ikr}$, (21) becomes

$$\frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} + k^2u = e^{ikr}f(r), \quad (22)$$

* Chap. I., Art. 5.
† Vortex strength is not vorticity, but proportional to the product of vorticity and sectional area of the vortex filament.
Now, if the equation
\[ \frac{d^2u}{dr^2} + P \frac{du}{dr} + Qu = 0, \]  
(23)
where \( P, Q \) are functions of \( r \), have independent integrals, \( u = \phi(r) \); \( u = \psi(r) \), the solution of
\[ \frac{d^3u}{dr^3} + P \frac{du}{dr} + Qu = F(r), \]  
(24)
may be written
\[ u = \int \frac{\phi(\rho) \psi(\rho) - \psi(\rho) \phi(\rho)}{\phi(\rho) \psi(\rho) - \psi(\rho) \phi(\rho)} F(r) \, d\rho. \]  
(25)
If we apply this formula in the present instance, making use of the relation
\[ I_n(x) K_n(x) - I_n(x) K_n(x) = 1/x, \]  
(26)
and choosing the arbitrary constants in (25) so that \( u \) vanishes when \( r = a, r = b, \) we again arrive at the equation (16).

**Art. 18. Application to Disturbance in which Initial Radial Velocity is \( \sin m(r - b) \sin kz \); with suitable values of the Constants, it increases greatly.**

Consider, then, a disturbance in which initially\(^*\)
\[ u = u_0 = \sin m(r - b) \sin kz, \]  
(27)
and therefore
\[ w = w_0 = (kr)^{-1} \{ \sin m(r - b) + m r \cos m(r - b) \} \cos kz, \]
where
\[ \sin m(a - b) = 0, \]
this value of \( u \) being the coefficient of \( i \) in \( \sin m(r - b) e^{ikz} \).

Here
\[ \{ r \left( k^2 + \frac{1}{\rho^2} \right) f(\rho) - \rho f(\rho) - \rho f''(\rho) \} e^{i(kz - \omega t)} \]
\[ = \{ [r \left( k^2 + m^2 \right) + 1/\rho] \sin m(r - b) - m \cos m(r - b) \} e^{i(kz - \omega t)}, \]  
(28)
and accordingly (16) gives, selecting the coefficient of \( i \), at time \( t \), on changing signs throughout,
\[ \{ I_n(ka) K_n(kb) - I_n(kb) K_n(ka) \} \sin k(z-Wt)[I_n(kp) K_n(kb) - I_n(kb) K_n(kp)] \]
\[ \times \int \left[ -[r(k^2 + m^2) + 1/\rho] \sin m(r - b) + m \cos (r - b) \right] \sin k(z-Wt)[I_n(kp) K_n(ka) - I_n(ka) K_n(kp)] d\rho \]
\[ + \{ I_n(ka) K_n(kb) - I_n(kb) K_n(ka) \} \sin k(z-Wt)[I_n(kp) K_n(kb) - I_n(kb) K_n(kp)] \]
\[ \times \int \left[ -[r(k^2 + m^2) + 1/\rho] \sin m(r - b) + m \cos (r - b) \right] \sin k(z-Wt)[I_n(kp) K_n(ka) - I_n(ka) K_n(kp)] d\rho, \]  
(29)
\* The example discussed in Arts. 20, 21 is analytically simpler.
where \( W = C\rho^3 + C' \). I wish to show that under certain conditions the value of \( u \) given by this equation may at some times and places become very great compared with its initial value.

Consider first the former of the two integrals in (29), and of it the portion which involves \( k^3 + m^3 \) as a factor, i.e. omitting this factor,

\[
\int_b^r \rho \sin m (\rho - b) \sin k (z - Wt) [I_i(kp)K_i(kb) - I_i(kb)K_i(kp)] d\rho. \quad (30)
\]

This may be written, as the difference of two, thus:

\[
\frac{1}{2} \left[ \int_b^r \rho \cos \{m (\rho - b) + k (z - Wt)\} [I_i(kp)K_i(kb) - I_i(kb)K_i(kp)] d\rho \right. \\
- \frac{1}{2} \left. \int_b^r \rho \cos \{m (\rho - b) - k (z - Wt)\} [I_i(kp)K_i(kb) - I_i(kb)K_i(kp)] d\rho. \right. \quad (31)
\]

Now the expression

\[
\rho \left[ I_i(kp)K_i(kb) - I_i(kb)K_i(kp) \right]
\]

is positive and increases continuously from zero as \( \rho \) increases from \( b \); for \( I_i(x) \) increases continuously with \( x \), as may be seen from its expansion in powers of \( x \), and \( xK_i(x) \) decreases continuously with increasing \( x \), as may be seen from the equation

\[
xK_i(x) = \frac{1}{2} \int_0^\infty e^{-\frac{1}{2}(a+2z/+)} dx. \quad (33)
\]

Moreover, when \( x \) is large, \( I_i(x), K_i(x) \) have the approximate values

\[
I_i(x) \approx \frac{1}{2}(2\pi x)^{1/2} e^x, \quad K_i(x) \approx x^{1/2}(2x)^{1/2} e^{-x}. \quad (34)
\]

It seems evident, then, that if \( kr \) is sufficiently large, the most important portion of the integrals (31), (32) is contributed by a small part of the range near the upper limit, and within which the approximate values given by (34) may be used. Considering first (31), making these approximations, and noting that if \( k (\rho - b) \) is large, \( I_i(kb)K_i(kp) \) is small compared with \( I_i(kp)K_i(kb) \), we replace (31) by

\[
(8\pi k)^{-1/2} K_i(kb) \int_b^r \rho^{3/2} e^{2b} \cos \{m \rho - mb + kz - kt (C\rho^3 + C')\} d\rho. \quad (35)
\]

In this, again, we may, without serious error, replace \( \rho^{3/2} \) by \( r^{3/2} \), so that under certain conditions it is approximately equal to

\[
(8\pi k)^{-1/2} r^{3/2} K_i(kb) \int_b^r e^{2b} \cos \{m \rho - mb + kz - kt (C\rho^3 + C')\} d\rho, \quad (36)
\]

and this again to

\[
(8\pi k)^{-1/2} r^{3/2} K_i(kb) \\
\times \left[ k \cos \{m \rho - mb + kz - kt (C\rho^3 + C')\} + (m - 2Ct/kp) \sin \{m \rho - mb + kz - kt (C\rho^3 + C')\} \right] \frac{e^{2b}}{k^2 + (m - 2Ct/kp)^2} \bigg|_b^r. \quad (37)
\]
and neglecting the value at the lower limit, we obtain

\[(8\pi k)^{-1} k^{1/2} K_0(kb) e^{\nu r} \]

\[\times \frac{k \cos \{mr - mb + kz - kt(Cr^2 + C') \} + (m - 2Ctkr) \sin \{mr - mb + kz - kt(Cr^2 + C') \}}{k^2 + (m - 2Ctkr)^2} \]  

(38)

I think it desirable to examine more carefully the validity of the approximations by which this value for (31) has been obtained, and to specify more fully conditions under which it may be used. It seems more convenient to consider the steps in the reverse order.

In the first place, when we come to take account of the second integral in (29), it will be found that the term in (38) which involves the sine is cancelled; and it is therefore a condition that \( \cos \{mr - mb + kz - kt(Cr^2 + C') \} \) is not too small a fraction.

The substitution of (38) for (37) is evidently justifiable if \( e^{\nu r-b} \) is large.

As regards the replacing of (36) by (37), if we denote by \( U \) the amount by which the integral in (36), considered as an indefinite integral, falls short of the approximate value substituted for it in (37), we evidently have

\[ e^{k_0} \frac{dU}{dp} = \frac{-2kCt \sin \{m_p - mb + kz - kWt\}}{k^2 + (m - 2Ctkp)^2} \]

\[+ 4kCt (m - 2Ctkp) [k \cos \{m_p - mb + kz - kWt\} + (m - 2Ctkp) \sin \{m_p - mb + kz - kWt\}] \]

\[\{k^2 + (m - 2Ctkp)^2\}^{3/2} \]  

(39)

The first term on the right is numerically less than \( 2Ct/k \), and the second than \( 4kCt (m - 2Ctkp)/(k^2 + (m - 2Ctkp)^2)^{3/2} \), and this again than \( 4kCt/[k^2 + (m - 2Ctkp)^2] \), and \( a \ fortiori \) less than \( 4Ct/k \). Thus by integration the value of \( U \) is certainly less than \( 6Ctc^{k_0r}/k^2 \). By using (38) when \( m - 2Ctkr \) is small compared with \( m \), and of order not greater than \( k \), \( U \) has been neglected in comparison with \( e^{k_0r}/k \), which is certainly legitimate if \( Ct/k \) is small, i.e., if \( m/k^2r \) is small.

Again, at such a time, the ratio of the difference between (35) and (36) to the final result, (38), is of the order of the ratio of

\[ \int_b^r (r^2 - r^1) e^{k_0} \cos \{m_p - mb + kz - kWt\} \, dp \]  

to \( r^ke^{k_0r}/k \).  

(40)

This integral is less than

\[ \int_b^r (r^2 - r^1) e^{k_0} \, dp \]

(41)

or, integrating by parts, than

\[ k^{-1} \left[ (r^2 - r^1) e^{k_0} + \int_b^r e^{k_0} r^{-1} \, dp \right] \]  

(42)
which, again, is less than

\[ (2k) \int_{0}^{r} e^{k \rho} d\rho. \]  

(43)

If \( kr \) is large, this is known to be approximately equal to

\[ e^{kr} / 2k^2 r^3, \]  

(44)

the neglect of which, in comparison with \( r^3 e^{kr} / k \), involves an error of order \( 1/kr \).

As regards the substitution of (35) for (31), the neglect of the term in (31) which involves \( I_1 (k \rho) K_1 (k \rho) \), is obviously valid. We have further replaced \( I_1 (k \rho) \) by \( (2\pi k \rho)^{-1} e^{k \rho} \), and the fractional error in so doing is known to be less than \( N / k \rho \) where \( N \) is a definite number. Thus, this approximation involves an error of order not greater than the neglect of \( \int_{0}^{r} e^{k \rho} d\rho / k \rho \) in comparison with \( r^3 e^{kr} / k \). The integral is less than \( k^{-1} \int_{0}^{r} e^{k \rho} d\rho \), which is known to be approximately equal to \( e^{kr} / 2k^2 r^3 \), and the error is thus of order not greater than \( 1/kr \).

To sum up, then, the substitution of (38) for (31), which it is intended to use, is legitimate when (i) \( m / k \) is large, (ii) \( k r \) is large compared with \( m / k \), (iii) the time is such that \( m - 2k \rho t \) is small and of order not larger than \( k \rho \), (iv) \( \cos (mr - mb + kz - Wtk) \) is not a very small fraction, (v) \( k (r - b) \) is large; and it will further be supposed below that (vi) \( k (a - r) \) is large.

I next proceed to show that under the same conditions the term (32) is negligible compared with (31). As before, by substituting the approximate values of the \( I, K \) functions, and neglecting in integration \( I_1 (k \rho) K_1 (k \rho) \) compared with \( I_1 (k \rho) K_1 (k \rho) \), we obtain the approximate value

\[ (8\pi k)^{-1} K_1 (k \rho) \int_{0}^{r} \rho \frac{\rho}{k} e^{k \rho} \cos \{m \rho - mb - kz + kt (C e^2 + C')\} d\rho, \]  

(45)

and this we replace by

\[ (8\pi k)^{-1} K_1 (k \rho) \int_{0}^{r} e^{k \rho} \cos \{m \rho - mb - kz + kt (C e^2 + C')\} d\rho. \]  

(46)

The approximations up to this point may be justified by reasoning similar to that which precedes.

Integrating by parts, the integral in (46) may be written

\[ \left| \frac{e^{k \rho} \sin \{m \rho - mb - kz + kt (C e^2 + C')\} \rho}{m + 2ktC e} \right|_{0}^{r} - \int_{0}^{r} \frac{e^{k \rho} \sin \{m \rho - mb - kz + kt (C e^2 + C')\} \rho}{m + 2ktC e} d\rho. \]  

\[ \times \left[ \frac{k}{m + 2Ctk \rho} - \frac{2ktC e}{(m + 2Ctk \rho)^2} \right] d\rho. \]  

(47)
At times such as considered the first term is of order $e^{\nu r}/m$. And the second is evidently less than (and, as a matter of fact, bears only a very small ratio to)

$$
\int_b^r e^{\nu \rho} \left[ \frac{k}{m + 2Ctkp} + \frac{2kCt}{(m + 2Ctkp)^2} \right] d\rho,
$$

(48)

which again is less than

$$
\int_b^r e^{\nu \rho} \left[ \frac{k}{m} + \frac{2kCt}{m^2} \right] d\rho.
$$

(49)

Remembering that the ratio of $kCt$ to $m$ is nearly unity, this is seen to be approximately equal to

$$
e^{\nu r} \frac{(1 + 2/k)}{m}.
$$

Accordingly, it appears that the neglect of (32) in comparison with (31) involves an error which, estimated as a fraction, is of order $k/m$, and which therefore is admissible.

We have still to consider the terms omitted from the first integral in (29).

The former of these, viz.:

$$
\int_b^r [I_1(kp)K_1(bb) - I_1(bb)K_1(kp)] \rho^{-1} m (\rho - b) \sin k(z - Wt) d\rho
$$

(50)

is less than

$$
\int_b^r I_1(kp)K_1(bb) \rho^{-1} d\rho;
$$

(51)

and from what has gone before, it is evident that, since $kr$ is large, this is approximately equal to

$$
(2\pi k^2 r^2)^{-1} e^{\nu \rho} K_1(bb);
$$

(52)

and the neglect of this, in comparison with the product of (38) by $k^2 + m^2$, at times such as considered, involves a fractional error of order $1/mr^2$.

The latter, viz.:

$$
\int_b^r [I_1(kp)K_1(bb) - I_1(bb)K_1(kp)] m \cos (\rho - b) \sin k(z - Wt) d\rho
$$

(53)

is less than

$$
mK_1(bb) \int_b^r I_1(kp) d\rho,
$$

(54)

i.e. than

$$
mK_1(bb) k^{-1} (I_1(kr) - I_1(bb)),
$$

(55)

which is approximately equal to

$$
(2\pi k^2 r^2)^{-1} m e^{\nu \rho} K_1(bb);
$$

(56)

and the neglect of this involves a fractional error of order $1/mr$. Thus, under the conditions stated, these terms may be neglected. The substitution of the product of (38) by $k^2 + m^2$ for the first integral in (29) has thus been justified.

Again, in the multiplier of the first integral in (29), we may omit $I_1(kr)K_1(ka)$ in comparison with $I_1(ka)K_1(kr)$, since $e^{\nu (\alpha - r)}$ is large, and on
substituting for the integral the product of (38) by \(+ k^2 + m^2\), there is obtained for the first term the approximate value

\[- I_\delta(\kappa a) K_\nu(\kappa r)(8\pi\kappa)^{1/2} 2K_\nu(\kappa + m^2) e^{\kappa r}\]

\[\times [k \cos(mr - mb + kz - k Wt) + (m - 2Ctkr) \sin(mr - mb + kz - k Wt)]/[(k^2 + (m - 2Ctkr)^2)], \]

or, replacing \(K_\nu(\kappa r)\) by its approximate value \(\pi^4(2kr)^{-1} e^{kr}\),

\[- I_\delta(\kappa a) K_\nu(\kappa b)(4k)^{-1}(k^2 + m^2)\]

\[\times [k \cos(mr - mb + kz - k Wt) + (m - 2Ctkr) \sin(mr - mb + kz - k Wt)]/[(k^2 + (m - 2Ctkr)^2)].\]

Taking next the second term in the right-hand member of (29), it may be proved by similar reasoning that under the same conditions, its approximate value differs from (58) only in having the sign of the coefficient of \(\sin(mr - mb + kz - k Wt)\) changed. Thus, by addition, noting that in (29) the second term in the coefficient of \(u\) is negligible compared with the first, and neglecting \(k^3\) compared with \(m^3\), there results

\[u \approx - m^2 \cos(mr - mb + kz - k Wt)/2 [k^2 + (m - 2Ctkr)^2]. \]

Caution is necessary in ascertaining from the equation of continuity the corresponding value of \(w\). Owing to the rapid rate at which the second term in the right-hand member of (29) varies with \(r\), this term may have to be taken account of. It may be shown that the approximate value of this term is obtainable from (59) by changing the sign of \(k\), and then changing the sign of the whole expression. Retaining the most important parts of each of the portions of \(w\), we obtain* 

\[w \approx m^2(m - 2Ctkr) \cos(mr - mb + kz - k Wt)/2k [k^2 + (m - 2Ctkr)^2] + m^2(m + 2Ctkr) \cos(mr - mb + kz + kWt)/2k [k^2 + (m + 2Ctkr)^2]; \]

or, substituting in the coefficient of the second cosine for \(2Ctkr\) its approximate value \(m^2\),

\[w \approx m^2(m - 2Ctkr) \cos(mr - mb + kz - k Wt)/2k [k^2 + (m - 2Ctkr)^2] + m \cdot (4k)^{-1} \cos(mr - mb - kz + kWt). \]

When \(m - 2Ctkr\) is of order \(k\), the former of the two terms of (60) is the more important; but when \(m - 2Ctkr\) is zero, the latter; it does not follow, though it may be shown to be true, that the second term is more important than the omissions from the first; it does follow, however, that when \(m - 2Ctkr\) is of order \(k\), \(w\) and \(u\) are of the same order of magnitude, but that when \(m - 2Ctkr\) is zero, \(w/u\) is small.

On comparing (59), (60) with (27), it is seen that, when \(m - 2Ctkr\) is of

* This deduction of the accurate value of \(w\) is not strictly justifiable. We ought to use the equation of continuity to obtain an accurate expression for \(w\) from that given for \(u\) by (29), and then approximate to its value.
order $k$, and subject to the other conditions stated, the value of $u$ will have increased so as to exceed its initial value in a ratio of order $m^3/k^4$. The initial value of $w$, however, exceeds that of $u$ in a ratio of order $m/k$, so that the kinetic energy averaged along a definite stream-line can increase in a ratio of order $m^3/k^2$ only.

In the preceding analysis, no supposition whatever has been made as to the value of $b$; and, consequently, by supposing it to diminish indefinitely, the results are applicable to the case of a complete pipe. It is, of course, only for a complete pipe that the steady motion here considered is the same as that which obtains in a viscous liquid.

We have here, then, an explanation of the observed instability. But the argument for instability, in the case of a disturbance of the type instanced, is weakened by the fact that the disturbance does not reach a maximum simultaneously for different values of $r$; in fact, the discussion goes to show that, at any particular time, it can be of the order of the maximum possible at the point considered only through a portion of the stream whose thickness is of order $kr/m$.

As affording some check on the accuracy of these results, it may be pointed out that, if we now further suppose that the ratio of $b$ to $a$ is made indefinitely near unity, we return to the problem discussed in the preceding chapter for the case in which, in the notation of that chapter, $lb$ is large; and it is easily verified that, under these suppositions, equations (59), (60) above agree with (38) of the preceding chapter, due allowance being made for the differences of notation. These differences are accounted for to a slight extent by my following Lord Rayleigh; unfortunately, however, I have introduced another discrepancy by choosing in the initial disturbance in one case the sine-function, and in the other the cosine, of the coordinate measured in the direction of flow.

**Art. 19. The Steady Motion is Stable for Sufficiently Small Initial Disturbance of the Type discussed.**

Moreover, the value of $u$ given by (29) eventually diminishes indefinitely as the time increases.

Writing the first of the two integrals in (29) in the form

$$\int_{\lambda} U \sin \{kz - kt (Cp^3 + C')\} \, d\rho;$$

and integrating by parts, it becomes

$$\left[ \frac{U \cos \{kz - kt (Cp^3 + C')\}}{2ktCp} \right]_{b}^{r} - \frac{1}{2ktCp} \int_{b}^{r} \cos \{kz - kt (Cp^3 + C')\} \frac{d}{d\rho} \left( \frac{U}{\rho} \right) \, d\rho. \quad (61)$$
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And it may be shown that the second term in this is eventually equal to
\[
\frac{1}{4k^2pC^2} \sin \{kz - kt(Cr^2 + C')\} \left[ \frac{1}{\rho} \frac{d}{d\rho} \left( \frac{U}{\rho} \right) \right]_{\rho=r}.
\] (62)

Now when the second integral in (29) is similarly treated, and both terms of (29) are combined, it will be seen that the terms which vary inversely as \( t \) cancel each other, and that the value of \( u \) thus eventually varies as \( t^2 \). But the equation of continuity shows that, this being so, the value of \( w \) eventually varies as \( t^1 \).

Thus for a disturbance of the type cited—and the argument is seen to apply equally to any ordinary disturbance which is periodic in the direction of flow, and symmetrical round the axis—the steady motion is stable, provided the initial disturbance is small enough, the kinetic energy of the relative motion eventually varying inversely as the square of the time; and this is true whatever the values of \( M, k, a, b \).

Art. 20. Disturbance in which Initial Radial Velocity is \( \sin m^2(r^2 - b^2) \sin kz \); with suitable values of the Constants it Increases Greatly.

As another example, consider a disturbance in which initially
\[
\begin{align*}
u &= \nu_0 = \sin m^2(r^2 - b^2) \sin kz, \\
w &= w_0 = (kr)^{-1} \left[ \sin m^2(r^2 - b^2) + 2m^2r^2 \cos m^2(r^2 - b^2) \right] \cos kz
\end{align*}
\] (63)

where \( \sin m^2(a^2 - b^2) = 0 \).

Here
\[
-\rho(k^2+1/p^3)f'(\rho)f''(\rho)+f'''(\rho)=-\left(4m^4p^4+k^2p+p^2\right)\sin m^2(p^2-b^2)\cos m^2(p^2-b^2);
\] (64)

and accordingly the right-hand member of this equation is to replace the first factor in the integrals of (29). Suppose \( ma \) large, and let us examine the value which this modified form of (29) gives for \( u \).

Consider first the former of the two integrals, \( \text{viz.} \), that whose range is from \( b \) to \( r \). It may be written as the difference of two, thus:
\[
- \frac{1}{2} \int_b^r \left[ (4m^4\rho^8 + k^2\rho + p^4) \cos \{ m^2 (p^2 - b^2) + kz + kt (Cr^2 + C') \} \right.
\]
\[
\left. + 4m^2p \sin \{ m^2 (p^2 - b^2) + kz + kt (Cr^2 + C') \} \right] 
\times [I_1 (kp) K_1 (kb) - I_1 (kb) K_1 (kp)] d\rho
\]
\[
- \frac{1}{2} \int_b^r \left[ (4m^4p^8 + k^2p + p^4) \cos \{ m^2 (p^2 - b^2) + kz + kt (Cr^2 + C') \} \right.
\]
\[
\left. + 4m^2p \sin \{ m^2 (p^2 - b^2) + kz + kt (Cr^2 + C') \} \right] 
\times [I_1 (kp) K_1 (kb) - I_1 (kb) K_1 (kp)] d\rho.
\] (65)

We will concern ourselves only with a time at which \( m^2 - kCt = 0 \). Taking
first the former of these integrals, at the time in question the angle whose cosine and sine occur does not involve \( \rho \).

Suppose that \( k(\rho - b) \) is large. It is not difficult to prove that when this is the case

\[
K_1(kb) \int_b^r \rho^n I_1(k\rho) \, d\rho = r^n I_1(kr) K_1(kb)/k,
\]

and that

\[
I_1(kb) \int_b^r \rho^n K_1(k\rho) \, d\rho \quad \text{is negligible in comparison.}
\]

Thus, \( k(r - b) \) and \( mr \) being large, if we further suppose that \( m^2r/k \) is large, the only term to be taken into account is that involving \( m^4\rho^5 \), and the approximate value of the first term of (65) is accordingly

\[
2m^4r^4 I_1(kr) K_1(kb) \cos(kz - m^3b^3 - kC't).
\]  

(66)

It may next be proved that, at the time in question, the second term of (65) is very small in comparison with (66), provided, of course, that the cosine which occurs in the latter is not a very small fraction. In this second term consider first the portion involving \( m^4\rho^3 \), i.e.:

\[
2m^4 \int_b^r \rho^3 \cos\{2m^2\rho^2 - m^3b^3 - kC't \} \left[ I_1(kp) K_1(kb) - I_1(kb) K_1(kp) \right] \, d\rho.
\]  

(67)

On integration by parts this may be written

\[
\frac{1}{2} m^2r^2 \left[ I_1(kr) K_1(kb) - I_1(kb) K_1(kr) \right] \sin\{2m^2\rho^2 - m^3b^3 - kC't \}
- \frac{1}{2} m^3 \int_b^r \sin\{2m^2\rho^2 - m^3b^3 - kC't \} \cdot \frac{d}{dp} \left[ \rho^3 I_1(kp) K_1(kb) - I_1(kb) K_1(kp) \right] \, d\rho.
\]  

(68)

The first term bears to (66) a ratio of order \( k/m^2r \). As regards the second term, the differential coefficient which appears in it is positive throughout the range, and consequently the integral is less, and, as a matter of fact, much less, than if the sine were replaced by unity, in which case it would be of the same order as the first term. Thus, the portion of the second term of (65) which involves \( m^4\rho^3 \) is negligible.

As regards the other portions of the second term of (65), each is, since

\[
I_1(kp) K_1(kb) - I_1(kb) K_1(kp)
\]

is positive, less than if the cosine or sine were replaced by unity, and even then they would be negligible in comparison with (66).

And this argument applies when \( b \) is zero, for division by the infinite \( K_1(kb) \) which then occurs in the left-hand member of (29), eliminates any disturbing infinity.

Thus, in (65), only the first term need be taken into account, and its approximate value is given by (66). This, then, is to be substituted for the
first integral in (29); and, as before, neglecting $I_i(\nu r)K_i(\nu a)$ in its multiplier, the first term of the right-hand member of (29) is replaced by

$$-\frac{2m^4r^3}{k}I_i(\nu a)K_i(\nu r)I_i(\nu r)K_i(\nu b)\cos(kz-m^2b^2-kC't),$$

(69)

or

$$-\frac{m^4r^3}{k^2}I_i(\nu a)K_i(\nu b)\cos(kz-m^2b^2-kC't).$$

(70)

And in a similar manner it may be shown that if $k(a-r)$ is large the second term also of the expression which replaces the right member of (29) is equal to (70). Adding, and dividing (29) by

$$I_i(\nu a)K_i(\nu b) - I_i(\nu b)K_i(\nu a),$$

in which the latter term is negligible, we obtain the approximate result

$$u = \frac{2m^4r^2}{k^2}\cos(kz-m^2b^2-mC't).$$

(71)

But, as in the case of the other disturbance, $w$ cannot, at this critical time, be found from this approximation; the portion of $u$ which involves the angle $(m^2+C't)^{1/2}-m^2b^2-kz+kC't$ is now more important for the determination of $w$.

It may be well to sum up here the suppositions made. They are that $k(a-r), k(a-r), mv, m^2r/k$ are each large, and that at the time $t$, to which these values apply, $m^2-Ckt = 0$.

A comparison of (71) with (63) shows that, as with the disturbance first instanced, the value of $u$ increases very much from the initial one, in a ratio of order $m^4r^2/k^2$ in fact. And, as before, the initial value of $w$ is much greater than that of $u$, so that the kinetic energy of the motion relative to the steady motion when averaged along a stream-line exceeds its initial value in a ratio of order $m^4r^2/k^2$, assuming that at the critical time $w$ is not of order larger than $u$.

It would seem that a disturbance of this latter type (63) is more unstable, or less stable, than that of the former type (27), as the critical time in the latter is the same at all points in the pipe; in fact, the values of the parameters which occur in the approximate equation (71) may be such that the equations are valid through a sufficient thickness of stream to render the kinetic energy of the relative motion through the whole pipe a very large multiple of its initial value.

If $k(a-b)$ is sufficiently large, equation (71) may indeed be used except through a very small fraction of the thickness of the stream adjacent to the walls. We may in this case obtain an approximate expression for the total relative kinetic energy. For this purpose we may introduce a stream function $\psi$ defined by the equations

$$ru = d\psi/dz, \quad rw = -d\psi/dr.$$
Denoting the relative kinetic energy by $T$, we have

$$T = \int \int r (u^2 + v^2) \, dr \, dz,$$

$$= \int \int \left( \frac{du}{dz} - \frac{dv}{dr} \right) \, dr \, dz,$$

$$= \int (\nu u - \lambda v) \psi \, dS - \int \psi \left( \frac{du}{dz} - \frac{dv}{dr} \right) \, dr \, dz,$$  \hspace{1cm} (72)

the former integral being taken over the bounding surfaces and $\lambda$, $\nu$ denoting the direction cosines of the normal. If the length of pipe included is a multiple of a wave-length, this integral is zero, since over the circular boundaries $\psi$ vanishes, and at the two plane ends the values of $\nu$ are identical, and the values of $\nu u - \lambda v$ equal, but of opposite signs. Thus, we have only to deal with the second integral, and in it $du/dr - dv/dr$ is seen from (19) to be of the form $f(z - Wt, r)$; this, of course, expresses that the vorticity flows with the stream; by reference to the initial conditions we have

$$du/dr - dv/dr = \left( \frac{4m^4r^2}{k} + k + \frac{1}{kr^2} \right) \sin \left( r^2 - b^2 \right) - \frac{4m^4}{k} \cos \left( r^2 - b^2 \right) \cos k(z - Wt).$$  \hspace{1cm} (73)

When this is transformed by expressing the product of two trigonometrical functions as a sum or difference, it is readily seen that at the critical time we have

$$\frac{du}{dz} - \frac{dv}{dr} = \left\{ \frac{2m^4r^2}{k} \sin \left( r^2 - b^2 \right) - \frac{4m^4}{k} \cos \left( r^2 - b^2 \right) \right\} \cos k(z - Wt).$$  \hspace{1cm} (74)

The most important term in $\psi$ again is seen from (71) to be

$$\psi = - 2m^4r^2/k^2 \sin \left( r^2 - b^2 \right) - m^3C'/C,$$  \hspace{1cm} (75)

Thus (72) gives as the average kinetic energy of the relative motion in the disturbance per unit length of pipe

$$2\pi m^4k^{-4} \int_b^a r^3 \, dr \quad \text{or} \quad \pi m^4 \left( a^4 - b^4 \right) (3k^{-1})^{-1}$$  \hspace{1cm} (76)

The corresponding expression initially is approximately

$$\pi m^4 \left( a^4 - b^4 \right)/4k^2.$$  \hspace{1cm} (77)

Thus, the energy in question is increased at the critical time from its initial value in a ratio which is approximately

$$4m^4 \left( a^4 + a^2b^2 + b^4 \right)/3k^2 \left( a^4 + b^4 \right),$$  \hspace{1cm} (78)

a ratio which is of the same order of magnitude as that of the value of $u^2$ at the critical time to the initial value of $w^2$; this proves, inter alia, that at the critical time the value of $w$ is of order at any rate not higher than that of $u$.

Here again, if we make the further supposition that the ratio of $b$ to $a$ is...
made indefinitely nearly unity, we revert to the problem of the preceding
Chapter for the case in which, in the notation of that Chapter, \( k_b \) is large, and
it is easily verified that then equation (71) above agrees with (38) of
Chapter I., and that the results deduced for the values of \( T \) at the critical
time agree also.

**Art. 21.** The Disturbance of Art. 20 increases greatly for other Relative Values
of the Constants.

In the case of the disturbance of type (63) the approximate values of
the velocities at the critical time may be obtained and similar conclusions
drawn for other relative values of the parameters than those stated above.

Suppose, instead, that \( mr, m^2r/k \) are large as before, but \( ka \), and therefore,
of course, also \( kr, k_b \) small. We now use the approximate values of the \( I, K \)
functions appropriate to small values of the argument, viz.:

\[
I_1(x) \approx x/2, \quad K_1(x) \approx 1/x. \tag{79}
\]

Considering the first term of (65), it is even easier than in the former
case to prove that the most important term in it is that involving \( m^4r^4 \); and
evidently at the time when \( m^2 - ktC \) is zero it is approximately equal to

\[
m^4b^{-1} \cos \left\{ kx - m^2b^3 - m^2C' \right\} \int_{b}^{r} \rho^2 (\rho^2 - b^2), \tag{80}
\]

that is, to
\[
m^4(3r^2 - 5b^2r^2 + 2b^4)/15b \cdot \cos \left\{ kx - m^2b^3 - m^2C' \right\}, \tag{81}
\]

or to
\[
m^4(r - b)^3(3r^2 + 6rb + 4r^2b + 2b^2)/15b \cdot \cos \left\{ kx - m^2b^3 - m^2C' \right\}; \tag{82}
\]

and at a point whose distances from the boundaries have a ratio neither very
large nor very small, this is of order \( m^4(a - b)/(a + b)^4b^2 \), provided, of course,
the cosine is not a very small fraction.

It may be proved also, that under these conditions, the second term of (65)
is negligible in comparison with (82). Consider first the portion of this
second term which involves \( m^4r^4 \); on substituting the approximate values of
the \( I, K \) functions, and the critical value of the time, it is seen to be nearly
equal to

\[
- m^4b^{-1} \int_{b}^{r} \rho^2 (\rho^2 - b^2) \cos \left\{ 2m^2 \rho^3 - m^2b^3 - kx + kC't \right\} d\rho, \tag{83}
\]

or, integrating by parts, to

\[
- \frac{1}{4} m^4b^{-1} \left( r^2 - b^2 \right) \sin \left\{ 2m^2 \rho^3 - m^2b^3 - kx + kC't \right\},
\]

\[
+ \frac{1}{4} m^4b^{-1} \int_{b}^{r} \left( 3 \rho^3 - b^2 \right) \sin \left\{ 2m^2 \rho^3 - m^2b^3 - kx + kC't \right\} d\rho. \tag{84}
\]

At a point such as referred to, the first term of this is of the order
\( m^2(a - b)(a + b)^3b^{-1} \), and therefore negligible compared with (82), provided
$m^2(a^2 - b^2)$ is large; and the second term, by replacing the sine by unity, is
seen to be of order not higher than the first (and, as a matter of fact, is much
smaller).

And, as with the former set of conditions, the remaining portions of the
second term of (65) are small compared with (66), and would be so even if
in them the sine or cosine were replaced by unity.

And, as before, the argument applies when $b$ is zero.

Thus, again in (65) only the first term need be taken into account; and its
approximate value is given by (82). This, then, is to replace the integral in
the first term of (29), and, substituting the approximate values of $I, K$, this
term becomes

$$- m^4(a^2 - r^2)(r - b)^4(3r^2 + 6r^2b + 4rb^2 + 2b^3)(30abr)^{-1} \cos \{kz - m^2b^2 - m^2C'/C\}. \quad (85)$$

And, in a similar manner, it may be shown that the second term of (29) is
replaced by a quantity differing from (85) only in having $a, b$ interchanged in
the multiplier of the cosine, and in having a plus sign prefixed.

Thus, by addition and subsequent division, the equation which replaces
(29) leads to the approximate result,

$$u = - m^4(a - r)(r - b) \times \{ (a + r)(r - b)(3r^2 + 6r^2b + 4rb^2 + 2b^3) + (r + b)(a - r)(3r^2 + 6r^2a + 4ra^2 + 2a^3) \} \times \{ 15r(a^2 - b^2)^{-1} \cos \{kz - m^2b^2 - m^2C'/C\} \}
\times \{ (a + b)r^2 + (a + b)r^2a + (a + b)r + ab(a^2 + ab + b^2) \} \times \{ 15r(a + b)^{-1} \cos \{kz - m^2b^2 - m^2C'/C\}. \quad (86)$$

Here, again, the value of $w$ cannot be found from this approximate expression
for $u$.

When $a - r$ and $r - b$ are not very unequal, this value of $u$ is of order
$m^4(a^2 - b^2)^2$; and its initial value is of order unity, so that it increases in a
ratio of this order. As in the other cases, the initial value of $w$, however,
exceeds the initial value of $u$, now in a ratio of order $m^4(a + b)/k$; thus, as
far as our investigation has gone, we cannot be sure that the disturbance
will increase much, unless $m^4(a - b)^3$ is large compared with $m^4(a + b)/k$,
or $m^4(a + b)k(a - b)^3$ is large, $k(a - b)$ itself being known to be small. This
condition for a large increase in the disturbance can, of course, be secured; but,
with the relative magnitudes chosen at the beginning of this Art., it
appears that at the critical time the value of $w$ is of order greater than $u$, and
that the additional condition just stated for a large increase is unnecessary.

We may, in fact, suppose that $ma$ is so large, and $ka$ so small, that (86) is
valid, except in comparatively small portions of the stream close to the walls,
and proceed as in Art. 20 to investigate the approximate value of the relative kinetic energy at the critical time. With the notation used therein,

$$T_T = - \int \psi \left( \frac{du}{dz} - \frac{dw}{dr} \right) dr dz,$$  \hspace{1cm} (87)

where $\frac{du}{dz} - \frac{dw}{dr}$ is given by (73), and its approximate value by (74). The approximate value of $\psi$, however, is not now as given by (75), but, as derived from (86), is

$$\psi \approx - 2m^4(a - r)(r - b)$$

$$\times \left\{ (a + b)r^3 + (a + b)^2 r^2 + (a + b)(a^2 + ab + b^2) r + ab(a^2 + ab + b^2) \right\} \{ 15k(a + b) \}^{-1}$$

$$\times \sin \{ kz - m^2 b^2 - m^2 C'/C \}.$$  \hspace{1cm} (88)

It seems unnecessary to evaluate the approximate expression for $T$, as it is somewhat complicated; evidently, however, it is of order

$$m^5 k^{-2} (a - b)^2 (a + b)^3,$$

whereas its initial value is of order

$$m^4 k^{-2} (a - b) (a + b)^2.$$

The ratio of the increase is thus of the order $m^4 (a^2 - b^2)^2$; and as the ratio of $u^2$ at the critical time to the initial $u^2$ has been shown to be of the smaller order $m^4 (a + b)^2 (a - b)^2 k^2$, it is evident that at the critical time $u$ is order higher than $v$, exceeding it in a ratio of order $[k(a - b)]^{-1}$.

Here, again, by way of verification, we observe that, if we now suppose the ratio $b/a$ to become indefinitely near unity, we revert to the problem of the preceding Chapter for the case in which, in the notation of that Chapter, $b$ is small; and it may be verified that under these circumstances the value of $u$, given by (86) above, agrees with that of $v$ given by (38), Chapter I., and that $\psi$ of (88) above agrees with $\psi$ of (44), Chapter I.

Another set of circumstances in which the propagation of the disturbance insisted above might be investigated in some detail is that in which $m(a - b)$ is large, and $ka$, $kb$ large, but $k(a - b)$ small. But from what precedes it is sufficiently evident that this cannot differ appreciably from the case just referred to of the principal problem of the preceding Chapter.

There seems no reason to suppose that the possibility of great increase is confined to disturbances of very great or very small wave-length in the direction of flow; the discussion of this Chapter deals in detail with cases only of one or other of these extreme types, for the reason that for them the formation of numerical estimates is less difficult.
CHAPTER III.

MOTION IN CYLINDRICAL STRATA ROTATING ROUND A COMMON AXIS.

ART. 22. Lord Rayleigh’s reference to this case.

Lord Rayleigh has remarked* that when the fluid is bounded by fixed concentric cylindrical walls, and the stream-lines are circles in planes perpendicular to the axis, the motion is stable, provided that in the steady motion the rotation continually increases or decreases from one boundary to the other.

As with the preceding cases of steady motion, he evidently refers to the fundamental disturbances solely (and even then, I think, the argument he indicates is inapplicable to those in three dimensions); but, as has been shown in the preceding chapters, stability for fundamental disturbances is quite compatible with instability for those of a more general character.

ART. 23. Two-dimensioned Disturbances when steady flow is that of Viscous Liquid; the Fundamental Types; Resolution of one initially arbitrary.

I proceed to discuss this problem also in some detail. Referring to two dimensions alone, we may conveniently use the current function \( \psi \), in terms of which the velocities in the disturbed relative to the steady motion are, radially \( u = d\psi/dr \theta \), and circumferentially \( v = -d\psi/dr \). If \( V \) denote the velocity in the steady motion, the vorticity is

\[
\frac{du}{rdr\theta} - \frac{1}{r} \frac{d}{dr} \{r(V + v)\},
\]

or

\[
\frac{d^2\psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} - \frac{1}{r^2} \frac{d\psi}{d\theta^2} + \frac{1}{r^2} \frac{d}{dr} \{rV\};
\]

and the differential equation governing the motion may be conveniently obtained by expressing that this remains constant for any given element of fluid, i.e. that

\[
\left( \frac{d}{dt} + (V + v) \frac{d}{rdr\theta} + u \right) \left( \frac{d^2\psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} + \frac{1}{r^2} \frac{d\psi}{d\theta^2} - \frac{1}{r} \frac{d}{dr} \{rV\} \right) = 0,
\]

or, if we retain only terms of the first order of small quantities

\[
\left( \frac{d}{dt} + V \frac{d}{rd\theta} \right) \left( \frac{d^2\psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} - \frac{d^2\psi}{\theta^2} \right) - \frac{u}{r} \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} (rV) \right) = 0, \tag{3}
\]

with the condition that \( \psi \) is to vanish at the boundaries.

If we were now to make \( \psi \) as a function of \( \theta \) and \( t \) vary as \( e^{i(\omega t+\theta)} \), it might be shown that the equation in \( \psi \), to which the boundary conditions lead, cannot be satisfied by a complex value of \( \psi \), if \( \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} (rV) \right) \) is one-signed throughout, which is evidently Lord Rayleigh's argument alluded to.

Whether we make this particular supposition or retain (3) in its most general form, it is evident that the equation is intractable, unless the velocity in the steady motion is such that

\[
\frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} (rV) \right) = 0,
\]

or

\[
V = Cr + C'r^3. \tag{4}
\]

This law, however, is that which applies in the case possessing the chief physical interest as being that which holds for viscous fluid when one or both of the bounding cylinders are made to rotate.

Taking this law, then, and supposing that \( \psi \) varies as \( e^{i(\omega t+\theta)} \), equation (3) becomes

\[
\left( n + \frac{V}{r} \right) \left( \frac{d^2\psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} - \frac{s^2}{r^2} \psi \right) = 0. \tag{5}
\]

The solution of this, subject to the given boundary conditions, resembles that of the preceding problems, in that it involves slipping in the interior of the fluid. If the outer and inner radii are \( a, b \), it is

\[
\psi = A \left( r^2a^2 - r^2b^2 \right)
\]

throughout a region adjoining the inner boundary, and

\[
\psi = B \left( r^2a^2 - r^2a^2 \right)
\]

through a region adjoining the outer, the surface of separation being that for which \( n + \frac{V}{r} \) is zero, and the coefficients \( A, B \) being so connected that the value of \( \psi \) is continuous. And it may again be proved that any disturbance of an ordinary type can be resolved into elements, each one of which is as described, by aid of the equation

\[
2s(r^2a^2 - r^2b^2)f(r) = \int_0^r (\rho^2b^2 - \rho^2a^2) \left( \rho f''(\rho) + f'(\rho) - s^2\rho^{-1}f(\rho) \right) d\rho
\]

\[
+ \int_r^\infty (\rho^2a^2 - \rho^2a^2) \left( \rho f''(\rho) + f'(\rho) - s^2\rho^{-1}f(\rho) \right) d\rho,
\]

provided \( f(a), f(b) \) are zero, and \( f(r), f'(r) \) are finite, continuous, and differentiable throughout the region. This equation may be discovered as before, and is easily verified.
And the result obtained is that, if originally \( \psi = f(r) \sin \theta \), then, at time \( t \),
\[
2s(\alpha^* + \alpha^* b^*) \psi
= \int_b^r \left[ p^* b^* \rho + \rho \rho''(\rho) + \rho'' f''(\rho) \right] \sin \theta \left( Vt/\rho \right) d\rho
+ \int_a^r \left[ \rho^* a^* \rho + \rho \rho''(\rho) + \rho'' f''(\rho) \right] \sin \theta \left( Vt/\rho \right) d\rho.
\]
the argument in \( V \) being \( \rho \).

And, again as before, this result may be otherwise obtained by noting that, the second member on the left-hand side of (3) being evanescent, a first integral of the equation is
\[
\frac{d^3 \psi}{dt^3} + \frac{1}{r} \frac{d \psi}{dr} + \frac{1}{r^3} \frac{d^3 \psi}{d \theta^3} = F(\theta - Vt/r, r),
\]
where \( F \) is a function determinate from the initial conditions, and in this instance equal to \( f(r) \sin \left( \theta - Vt/\rho \right) \); then equation (8), integrated subject to the conditions that \( \psi \) vanishes for \( r = a \), \( r = b \), will be found to lead to (7).

**Art. 24. Motion is Stable for a sufficiently small Initial Disturbance varying as \( \sin s\theta \).**

As with the previous problems, if \( f(r) \) be any function of an ordinary type, the motion is stable for the disturbance given initially by \( \psi = f(r) \sin \theta \), **provided the initial value is small enough.** For, if we denote the first integral in the right-hand member of (7) by
\[
\int_b^r U \sin \left[ \theta - (C + C' \rho^2) t \right] d\rho,
\]
on integration by parts this may be written in the form
\[
-(2sC't^3 - r^2 U \cos \left[ \theta - (C + C' \rho^2) t \right] + (2sC't^3) \int_b^r \cos \left[ \theta - (C + C' \rho^2) t \right] \frac{d}{d\rho} (\rho^2 U) d\rho,
\]
the second term of which may be shown to be eventually of order \( t^3 \). When the integral in the second term of the right-hand member of (7) is treated in a similar fashion, and the two terms combined, it is evident that in the resulting expression for \( \psi \) the terms of order \( t^4 \) cancel, and that \( \psi \) is eventually of order \( t^2 \). Thus, as is seen by differentiating \( \psi \), the radial velocity ultimately varies as \( t^2 \), and that in the direction of flow as \( t^4 \).

This argument applies even when \( a \) is increased indefinitely (in which case \( C \) is zero, otherwise the velocity in steady motion would be infinitely great at infinity).

It does not apply, however, if \( C' \) is zero, that is, if the fluid rotates like a rigid body; in this case (7) shows that the disturbance neither increases nor
decreases, but is simply carried round by the fluid, the velocity of each element remaining invariable.

**ART. 25. Disturbance having Stream-Function initially** \( \sin \epsilon (r^2 - b^2) \sin \theta \); for suitable Constants it increases greatly.

And as in the steady motions discussed in the previous chapters, we may show in the case of some disturbances of analytically simple types that the disturbance before dying out will increase, and increase very much.

Consider a disturbed motion in which initially

\[
\psi = \psi_0 = \sin \epsilon (r^2 - b^2) \sin \theta \quad \text{where} \quad \sin \epsilon (r^2 - b^2) = 0.
\]

Here

\[
\rho f''(\rho) + f'(\rho) + \xi^2 \rho^3 f(\rho) = 4\epsilon^2 \rho^2 \cos \epsilon (r^2 - b^2) - (4\epsilon^2 \rho^2 + \xi^2 \rho^2) \sin \epsilon (r^2 - b^2);
\]

and accordingly we have at time \( t \)

\[
2\epsilon (\epsilon^2 b^4 - \epsilon^2 a^4) \psi = (\epsilon^2 a^4 - \epsilon^2 b^4)
\]

\[
\times \int_b^\infty \left( \rho^2 b^2 - \rho^2 a^2 \right) \left( 4\epsilon^2 \rho^2 \cos \epsilon (r^2 - b^2) - (4\epsilon^2 \rho^2 + \xi^2 \rho^2) \sin \epsilon (r^2 - b^2) \right) \sin \theta \left( C + \epsilon \rho^2 \right) \rho^2 \left( \rho \right) \, d\rho.
\]

We will obtain the approximate value of \( \psi \) as given by this equation at the time when \( \epsilon = \alpha \epsilon \). Consider the first integral in the right-hand member; it may be expressed as the difference of two, thus:

\[
\frac{1}{2} \int_b^\infty \left( \rho^2 b^2 - \rho^2 a^2 \right) \left( 4\epsilon^2 \rho^2 \sin \theta \left( C - \epsilon \rho^2 \right) \rho^2 \left( \rho \right) + (4\epsilon^2 \rho^2 + \xi^2 \rho^2) \sin \theta \left( C + \epsilon \rho^2 \right) \rho^2 \left( \rho \right) \right) \, d\rho
\]

\[
- \frac{1}{2} \int_b^\infty \left( \rho^2 b^2 - \rho^2 a^2 \right) \left( 4\epsilon^2 \rho^2 \sin \theta \left( C + \epsilon \rho^2 \right) \rho^2 \left( \rho \right) + (4\epsilon^2 \rho^2 + \xi^2 \rho^2) \sin \theta \left( C - \epsilon \rho^2 \right) \rho^2 \left( \rho \right) \right) \, d\rho.
\]

At the time referred to, the former of these, or

\[
\frac{1}{2} \sin \left( \theta - C \epsilon \rho^2 \right) \left( 4\epsilon^2 \rho^2 \sin \theta \left( C - \epsilon \rho^2 \right) \rho^2 \left( \rho \right) \right) \, d\rho
\]

\[
+ \frac{1}{2} \cos \left( \theta - C \epsilon \rho^2 \right) \left( 4\epsilon^2 \rho^2 \sin \theta \left( C - \epsilon \rho^2 \right) \rho^2 \left( \rho \right) \right) \, d\rho,
\]

is equal to

\[
2\epsilon \sin \left( \theta - C \epsilon \rho^2 \right) \left[ \frac{r^2 \rho^2}{s - 2} + \frac{r^{s-2} \rho^2}{s + 2} \right] - \frac{2s \rho^2}{s^2 - 4} \]

\[
+ \frac{1}{2} \cos \left( \theta - C \epsilon \rho^2 \right) \left[ 4\epsilon \left( \frac{r^{s-1} \rho^2}{s - 4} + \frac{r^{s-1} \rho^2}{s + 4} \right) - \frac{2s \rho^2}{s^3 - 16} + s \left( r^2 \rho^2 + r^2 \rho^2 \right) \right].
\]
The second integral in (11) may also be expressed as the difference of two, of which one differs from (14) only in having \( b \) replaced by \( a \), and in having the opposite sign.

When these terms,* one from each of the integrals in (11), are combined as in (11), the resulting contribution to the right-hand member of that equation may be written in the form

\[
2s \left[ \frac{2c^4}{s^4 - 4} \begin{pmatrix} r^s & a^2 & b^2 \\ r^s & a^2 & b' \\ r^s & a^2 & b'' \end{pmatrix} \sin \{s\theta - Cc'/C' - c'b^2\} + \begin{pmatrix} r^s - a^2 & b^4 \\ r^s & a^2 & b' \\ r^s & a^2 & b'' \end{pmatrix} \right] \cos \{s\theta - Cc'/C' - c'b^2\} \right]. \quad (15)
\]

Not much information can be obtained from this without making some definite supposition as to the relative values of \( a, b, r \). If, for instance, we suppose that \( b/a \) is nearly unity, it is evidently to be anticipated that the results obtainable can differ but little from those which hold in the case of the chief problem discussed in Chapter I.

As another case in which results may be expressed with sensible accuracy in a comparatively simple form take that in which \( b'/r, r'/a' \) are both small.

The determination of the most important terms in the determinants in (15) depends further on the value of \( s \). If we suppose \( s \) large, the most important terms in each are in order

\[
- a'b^s r^s, \quad - a^b r^s, \quad - a^b r^s,
\]

so that the approximate value of (15) under these circumstances is

\[
- 2ao^s b^s \left[ \frac{2c^4}{s^4 - 4} \sin \{s\theta - Cc'/C' - c'b^2\} + \left( \frac{2c^4}{s^4 - 16} + 1 \right) \cos \{s\theta - Cc'/C' - c'b^2\} \right], \quad (16)
\]

and if we now further suppose \( c'r^s s^s \) large, this is sensibly equal to

\[
- 4a^b r^s \cos \{s\theta - Cc'/C' - c'b^2\}; \quad (17)
\]

this result is equivalent to the replacing of (14) by the solitary term

\[
\frac{2c^4 r^s}{s^4 - 4} \cos \{s\theta - Cc'/C' - c'b^2\}, \quad (18)
\]

and the treating of the second integral in (11) similarly.

It may next be shown that, with the conditions stated, the other terms are negligible, which would be omitted in thus replacing the right-hand member of (11) by (16). In the integral which occurs in the first term in this right-hand member we have omitted the second term of (12). It may

---

* I.e. (14) and the analogue obtained from it by changing \( b \) into \( a \), and changing sign.
be proved that the most important part of this, at the critical time when $c^3 - sCt$ vanishes, is

$$- \int_b^r (\rho^3 t^3 - \rho^3 t^3) \ 4c^4 r^3 \cos(2c^3 r^3 + c^2 C/C - c^3 b^3 - s\theta) \ d\rho,$$

or, integrating by parts,

$$4c^d (r^3 t^3 - r^3 t^3) \ r^7 \sin(2c^3 r^3 + c^2 C/C - c^3 b^3 - s\theta)$$

$$- \frac{d}{dp} \int_b^r \sin(2c^3 r^3 + c^2 C/C - c^3 b^3 - s\theta) \ d(p^3 t^3 - p^3 t^3) \ d\rho.$$  (20)

It is readily seen that the contribution of the first term in this to the right-hand member of (11) is cancelled by a similar expression of opposite sign which arises when the second term of (11) is treated in a similar fashion. The second term in (20) is less—and, as a matter of fact, much less—than if the sine were replaced by unity and

$$\frac{d}{dp}((\rho^3 t^3 - \rho^3 t^3) \ r^3);$$

by its numerical value; and as

$$(\rho^3 t^3 - \rho^3 t^3) \ r^3,$$

continually increases from zero as $\rho$ increases from $b$, it would then be equal to

$$- \frac{d}{dp} (\rho^3 t^3 - \rho^3 t^3) \ r^3,$$

and the ratio of this to (18) is, unless the cosine in (18) be a very small fraction, of order $sr^2 c^3$, which we have supposed small.

In a similar manner it may be proved that the omissions which have been made from the second term of the right-hand member of (11) in replacing that member by (17) are legitimate.

Making this substitution, (11) is thus sensibly equivalent to

$$\psi = \frac{2c^r r^3 s^3 \cos(s\theta - c^2 C/C' - c^3 b^3)}{2c^r r^3 s^3 \sin(s\theta - c^2 C/C' - c^3 b^3)},$$

which exceeds its initial value in a ratio of order $c^r r^3 s^3$, which has been supposed large.

At the critical time the radial velocity

$$u = d\psi/d\theta = \frac{2c^r r^3 s^3 \sin(s\theta - c^2 C/C' - c^3 b^3)}{2c^r r^3 s^3 \cos(s\theta - c^2 C/C' - c^3 b^3)}$$

exceeds its initial value in a ratio of the same order, $c^r r^3 s^3$.

But, as in the problems of the preceding chapters, the velocity in the direction of flow cannot, at least prima facie, be obtained by differentiating (22), for the reason that in this equation there has been neglected a term involving the angle

$$2c^r r^3 s^3 \sin(s\theta - c^2 C/C' - c^3 b^3 - s\theta),$$

and differentiation with respect to $r$ introduces a relatively large multiplier.

By differentiating equation (11) it may be shown that at the critical time
the relative velocity in the direction of flow bears to the radial velocity a ratio of order \(1/s\).

Thus, as the initial relative velocity in the direction of flow exceeds the relative velocity in a ratio of order \(\sigma r^2 s^{-1}\), the increase in the resultant relative velocity is of order \(\sigma r^2 s^{-1}\).

We may, as with the previous problems, use our results to obtain approximately the kinetic energy of the relative motion. If \(T\) be the amount of this energy for unit length of the cylinder, we have

\[
2T = \int_{\alpha}^{\beta} r (u^2 + v^2) \, dr \, d\theta
\]

\[
= \int_{\alpha}^{\beta} \int_{0}^{\pi} \left( u \frac{d\psi}{d\theta} - rv \frac{d\psi}{dr} \right) \, dr \, d\theta
\]

\[
= - \int_{\alpha}^{\beta} \left( \psi v \right)_{\theta} + \int_{0}^{\pi} \left( \psi v \right)_{\theta} - \int_{\alpha}^{\beta} \psi \left( \frac{d}{dr} (rv) \right) dr + \frac{d\psi}{d\theta} \cdot \frac{d\psi}{d\theta}.
\]

(23)

The first and second terms vanish since \(\psi\) is zero along the bounding cylinders. As regards the remaining term, the value of \(\psi\) is given approximately by (22), and the value of

\[
du/d\theta - d(rv)/dr \quad \text{or} \quad \psi v/dr^2 + 1/r \cdot \psi v/d\theta^2
\]

is given accurately, i.e., as far as the first powers of small terms, by an equation of type (8); and in this case is, (see (10)),

\[
\{4c^2 \sigma^2 \cos \theta (r^2 - b^2) - (4c^2 \sigma^2 + s^2 r^2) \sin \theta (r^2 - b^2) \} \sin \frac{\theta}{C + C' r^2} d\theta.
\]

(24)

Replacing the product of two trigonometrical functions by a sum or difference, and substituting the critical value of the time, viz. \(t = \psi/(sC')\), this may be written as the difference of two expressions, thus:

\[
\frac{1}{2} \{4c^2 \sigma^2 \sin \psi \theta - c^2 C'(C - c^2 b^2) + (4c^2 \sigma^2 + s^2 r^2) \cos \psi \theta (C - C' r^2 - c^2 b^2) \}
\]

\[- \frac{1}{2} \{4c^2 \sigma^2 \sin (2c^2 r^2 + c^2 /C' (C - c^2 b^2 - s\theta)) + (4c^2 \sigma^2 + s^2 r^2) \cos (2c^2 r^2 + s^2 /C' (C - c^2 b^2 - s\theta)) \}.
\]

(25)

Evidently, in multiplying this by (22), in order to find the integral which constitutes the final term of (23), we may neglect the second of these two expressions, owing to its rapid fluctuations with respect to \(r\); and thus, performing the integral with respect to \(\theta\), we have the approximate result

\[
2T \approx \pi c^2 s^{-1} \int_{\beta}^{\alpha} (4c^2 r^2 + s^2 r^2) \, dr \approx \frac{1}{2} \pi c^2 b^2 s^{-1},
\]

(26)

provided \(c b^2 s^{-1}\) is large.

The initial value of \(T\), obtainable in the same manner or otherwise, is given approximately by

\[
2T_0 \approx \pi \int_{\alpha}^{\beta} (4c^2 r^2 + s^2 r^2) \, dr \approx \frac{1}{2} \pi c^2 b^2.
\]

(27)

Thus the kinetic energy increases from its initial value in a ratio of order
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$c^t b^t s^t$ which has been supposed large; if the supposition, made early in the investigation, that $c^t r^t s^t$ is large, should not hold up to the outer boundary $r = a$, the discussion yet suffices to show that the relative kinetic energy throughout the region for which the supposition does hold will be much increased; the fact that $r^t/a^t$ and $b^t/r^t$ cannot both be small close to a boundary does not, however, sensibly affect the result if $s$ be sufficiently large.

If, instead of taking $s$ large, as in the preceding, we consider the case in which $s$ is unity, we may show that if $a/r$, $r/b$, $c^t r^t$ are each large, then at the critical time the radial velocity bears to its initial value a ratio of order $c^t r^t$, at that time the radial and circumferential (relative) velocities are of the same order of magnitude, and the resultant (relative) velocity bears to its initial value a ratio of order $c^t r^t$.

The single type of disturbance, which is investigated above, appears sufficient to illustrate the possibility of instability.
ERRATA.

SECTION A.

Page 15, line 8, for \( m \) read \( m^2 \)

,, ,, 3 from bottom, for a read any

,, 20, ,, 22, for 3B read 3A

,, 25, ,, 25, for B read \( B \)

,, 30, ,, 19, for \( \lambda/m \) read \( m/\lambda \)

,, 31, ,, 19, for \( \xi \) read \( \xi \)

,, 35, ,, 17, for \( \xi \) read \( \xi \)

,, 17, last line, for \( (\sqrt{5} - 1) \) read \( (\sqrt{5} - 1)/2 \)