Time-dependent, coupled, Ekman boundary layer solutions incorporating Stokes drift

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Abstract

Recent technological advances in current measuring devices has resulted in a large observational database related to wind-driven motions in the upper ocean mixed layer. This has served to highlight the fact that transient motions make up a substantial contribution of the resulting Ekman currents. At the same time, certain discrepancies have emerged between the observed angular deflections of the steady-state currents from the surface wind stress, both at the surface and at sub-surface depths, which cannot be reconciled using the classical Ekman model. This paper seeks to tackle these two issues.

First a general analytical method is presented for solving the time dependent horizontal momentum Ekman equations. Analysis of the unsteady terms that arise from simple special cases shows how the evolution proceeds through three stages. At early times, the Coriolis acceleration is insignificant, and the current is unidirectional and deepens through downward diffusion of momentum. Later Coriolis acceleration deflects the current vectors in the upper layers, whilst downward diffusion of momentum continues to deepen the layer. Finally, once diffusion has penetrated down to the depth of the steady-state current, then the transients decay on the inertial or diffusive timescale, depending upon the boundary conditions of the particular problem.

In the second half of the paper, a new steady-state model is developed that includes the effects of wind-generated waves, through the action of their Stokes drift on the planetary vorticity. Comparisons between observations and the theoretical predictions, demonstrate that inclusion of the Stokes drift is the key to reconciling the discrepancies in the angular deflections of the steady-state currents. This leads to the conclusion that Ekman layer currents are significantly influenced by the surface waves.

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Keywords: Ekman currents; Stokes drift; Transients; Inertial oscillations; Angular deflections

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1. Introduction

Ekman (1905) was the first to explain, quantitatively, the observation that the flow of steady wind-driven ocean currents are deflected to the right of the prevailing wind direction (in the northern hemisphere). This is due to the momentum balance between the Coriolis acceleration acting on the upper ocean layers and the frictional forces generated by the turbulent stress imposed by the wind. Subsequently, the study of the Ekman boundary layer has been the subject of considerable theoretical research, e.g. Munk (1950), Thomas (1975), Madsen (1977), Huang (1979) and Price and Sundermeyer (1999). Observed upper ocean currents are typically made up of a complex superposition of surface waves, plus wind-driven (Ekman), tidal and pressure-driven (geostrophic) flows. Estimating the size of the relative contributions is difficult and consequently direct experimental evidence concerning the structure and profile of the Ekman boundary layer has, until recently, been lacking. However, new measuring devices e.g. Weller and Davis (1980), together with more powerful computer resources, means that more detailed and reliable measurements of wind-driven currents are now available, e.g. Churchill and Csanady (1983), Price et al. (1987), Krauss (1993), Wijffels et al. (1994), Niiler and Paduan (1995) and Chereskin (1995).

The advent of this observational database has highlighted a number of features, which have up to now attracted relatively little comment:

(i) Modern fast-response measurement techniques, which can give a reading every 2 s (e.g. Weller and Plueddemann, 1996) result in signals containing a substantial component derived from inertial oscillations, together with variations due to alterations in the wind stress, often superimposed on an approximately steady-state mean-flow. As the sophistication of instrumentation improves such datasets will become more common, and their characteristics, in particular the nature of the unsteady terms, are likely to be of increasing interest. However, apart from the Fredholm solution derived for a constant (laminar flow) vertical eddy viscosity $\nu_e$, published by Ekman (1905) in his pioneering work, the unsteady Ekman problem has received scant attention (see also Madsen, 1977). Although the advent of high speed computing resources renders the calculation of the numerical solution of the most sophisticated unsteady Ekman problem routine, the nature of the underlying time-dependent flow is difficult to discern. The characteristics of such a flow can be illustrated by studying the analytical solutions derived for relatively simple Ekman type problems. Such solutions will be derived in this paper.

(ii) Measurements of the steady-state mean current, derived from long-term time-averaged observations, reveal three features that need to be explained. Firstly the surface current lies at an angle 10–45° to the wind stress, e.g. Smith (1968) and Huang (1979). Secondly, the sub-surface current (here sub-surface is defined to mean depths from ~5 to 20 m below the surface) is deflected by approximately 75° from the wind stress (Price and Sundermeyer, 1999). Thirdly, the current is rapidly attenuated below the surface. Ekman’s (1905) laminar model predicts a surface deflection of exactly 45°, but the associated sub surface currents die off far too slowly to provide a realistic representation of the observed results. In practice the ocean surface acts as a barrier, limiting the characteristic size of the turbulent eddies which drive the water motion at shallow
Table 1

Typical current values and deflection angles derived for the coupled oceanic–atmospheric Ekman log-layer solution (30), assuming the surface stress $\tau_S = 0.09 \text{ N m}^{-2}$, windspeed $U_{10} = 7 \text{ m s}^{-1}$ and Coriolis frequency $f = 10^{-4} \text{ s}^{-1}$.

<table>
<thead>
<tr>
<th>$q$ (%)</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_G$ (m s$^{-1}$)</td>
<td>9.5</td>
<td>9.5</td>
</tr>
<tr>
<td>$u_A$ (ms$^{-1}$)</td>
<td>0.28</td>
<td>0.28</td>
</tr>
<tr>
<td>$u_S$ (m s$^{-1}$)</td>
<td>$9.5 \times 10^{-2}$</td>
<td>$9.4 \times 10^{-2}$</td>
</tr>
<tr>
<td>$z_A$ (m)</td>
<td>$4.0 \times 10^{-4}$</td>
<td>$4.1 \times 10^{-4}$</td>
</tr>
<tr>
<td>$z_S$ (m)</td>
<td>0.47</td>
<td>2.4 $\times 10^{-2}$</td>
</tr>
</tbody>
</table>

Depth $z$ (m) of current $U$

| Current magnitudes and deflection angles from surface stress |
|---|---|---|
| Surface | $0.0838 \text{ m s}^{-1}$ at 21.8$^\circ$ to $\tau_S$ | $0.149 \text{ m s}^{-1}$ at 13.8$^\circ$ to $\tau_S$ |
| 5 | $0.0356 \text{ m s}^{-1}$ at 46.5$^\circ$ to $\tau_S$ | $0.0368 \text{ m s}^{-1}$ at 48.1$^\circ$ to $\tau_S$ |
| 10 | $0.0258 \text{ m s}^{-1}$ at 59.0$^\circ$ to $\tau_S$ | $0.0257 \text{ m s}^{-1}$ at 61.0$^\circ$ to $\tau_S$ |
| 20 | $0.0162 \text{ m s}^{-1}$ at 76.6$^\circ$ to $\tau_S$ | $0.0162 \text{ m s}^{-1}$ at 78.9$^\circ$ to $\tau_S$ |
| 30 | $0.0117 \text{ m s}^{-1}$ at 90.0$^\circ$ to $\tau_S$ | $0.0117 \text{ m s}^{-1}$ at 92.5$^\circ$ to $\tau_S$ |

depths. Consequently only depth-dependent parameterisations of $\nu_e$ are likely to produce realistic models of wind-driven ocean currents. One relatively simple model was formulated by Madsen (1977) in which $\nu_e$ was assumed to vary linearly with depth. This led to a plausible, although somewhat low, predicted surface current deflection angle of about 10$^\circ$. However, the predicted sub-surface deflection angles are much less than the 75$^\circ$ value observed (see Table 1 and subsequent discussion in this paper). Clearly, new theoretical ideas are needed to reconcile all these three features of the observed steady-state Ekman current.

The purpose of this paper is to tackle these two issues. Firstly, a general analytical method is presented for solving the unsteady Ekman boundary layer equations. For illustrative purposes, a number of specific solutions will be discussed for relatively simple Ekman problems. These analyses highlight the nature of the resulting unsteady terms, the importance of the applied boundary conditions and the decay of transient motions.

Secondly, the characteristics of the resulting steady-state terms are examined with a view to answering point (ii) raised above. The Ekman solution, obtained using a linear variation of $\nu_e$ with depth, is built upon in two ways. Specifically, the oceanic Ekman
layer is coupled to a corresponding atmospheric boundary layer above the sea surface. This allows the value of the eddy viscosity to be estimated in terms of observable physical quantities such as wind speed and surface wind stress. In addition, it is postulated that the observations show manifestations of wave related processes, in particular the influence of Stokes drift. Stokes drift can substantially affect the whole of the mixed layer in two ways. First, it deforms the vorticity associated with the mixed layer turbulence generating Langmuir circulations (Craik and Leibovich, 1976; Leibovich, 1977a,b, 1983; Teixeira and Belcher, 2002). These streamwise vortices enhance the local turbulent kinetic energy (Skyllingstad and Denbo, 1995; McWilliams et al., 1997) and are believed to regulate the depth of the mixed layer. Second, it deforms the planetary vorticity via a modified Coriolis force term in the linear horizontal momentum equations, altering the balance of the mean flow in the mixed layer. McWilliams et al. (1997) examined the steady-state solutions of this ‘Stokes–Ekman’ boundary layer model, assuming a constant eddy viscosity. Here, the corresponding equations are solved assuming $v_e$ varies linearly with depth. The results suggest that it is the Stokes drift which is key in reconciling the various characteristics of the measurements.

The layout of the paper is as follows. Section 2 briefly establishes the classical Ekman equations governing wind-driven currents, subject to various boundary conditions. Section 3, together with the mathematical Appendices A–D, presents the time-dependent solutions for both the laminar flow and the Madsen (1977) eddy viscosity model. Section 4 presents time-dependent results of the coupled oceanic–atmospheric Ekman system when $v_e$ varies linearly with depth. Section 5 makes some comparisons with some field observations. In Section 6, the implications of these results leads to the incorporation a Stokes drift term into the coupled system. Finally, some concluding remarks are presented in Section 7.

### 2. The mathematical model

For a deep, vertically homogeneous ocean, of infinite lateral extent, the horizontal momentum equations take the form:

\[ \frac{\partial U_O}{\partial t} + i f U_O = \frac{1}{\rho} \frac{\partial}{\partial z} \tau - \frac{1}{\rho} \nabla p + \text{higher-order terms}. \]  

(1)

Here, $U_O(z, t) = U_O + i V_O$ is the complex horizontal velocity in the $x$-$y$ plane, $f$ is the Coriolis parameter, $\rho$ is the density of sea water, $\nabla p = (\partial p/\partial x + i \partial p/\partial y)$ the horizontal pressure gradient, and $\tau(z, t) = \tau_x + i \tau_y$ is the shear stress due to molecular and turbulent processes. The higher-order terms, representing correlations between the various variables, are presumed to be small. The observed horizontal velocity is now decomposed into pressure-driven (geostrophic) and wind-driven (Ekman) components $U_O = U_p + U$ such that

\[ \frac{\partial U_p}{\partial t} + i f U_p = -\frac{1}{\rho} \nabla p, \]  

(2a)

\[ \frac{\partial U}{\partial t} + i f U = \frac{1}{\rho} \frac{\partial}{\partial z} \tau. \]  

(2b)
The stress term in (2b) is governed by a turbulent momentum flux $\tau = -\rho(\langle U'_O x + i V'_O y \rangle W)$, where $(U'_O x, t), (V'_O y, t), (W, t)$ denotes the fluctuating component of the turbulent velocity field and (7) represents time and spatial averaging over a given $x-y$ plane.

Following Ekman (1905), the effects of the ocean turbulence, Gargett (1989), and the influence of surface waves, e.g. Craig and Banner (1994) and Terray et al. (1996) are summarised by an eddy viscosity linking the applied stress to the mean velocity gradient, namely

$$\tau = \rho \nu_e(z) \frac{\partial U}{\partial z}. \quad (3)$$

Combining (2b) and (3) gives the Ekman equations for the wind-driven ocean currents

$$\frac{dU}{dt} + ifU = \frac{\partial}{\partial z} \left( \nu_e(z) \frac{\partial U}{\partial z} \right). \quad (4)$$

Solutions to (4) can be obtained subject to the appropriate boundary and initial conditions. The ocean is assumed to be initially at rest $U = 0, t \leq 0$. Then, for times $t > 0$, a steady uniform wind blows and applies a uniform constant wind stress $\tau_S$ to the ocean surface, $z = 0$. Hence, the surface boundary condition is

$$\rho \left( \nu_e(z) \frac{\partial U}{\partial z} \right) \bigg|_{z=0} = \tau_S, \quad \text{at} \quad z = 0. \quad (5)$$

In this paper, it will be assumed that $\tau_S$ is constant and, without loss of generality, is directed along the positive $x$-axis, i.e. $\tau_S = \tau_S + i0$. (The solution methods outlined in the next section can be generalised to accommodate a time varying-wind stress, see also Madsen (1977).)

The lower boundary condition is specified in three ways. Firstly, the currents are taken to vanish deep in the ocean, i.e. $U \to 0$ as $z \to \infty$. In the subsequent analysis, it will be demonstrated that the nature of the time-dependent terms in the solutions depends upon on the lower boundary condition. Hence, it will prove useful secondly, to look at the cases when $U = 0$ at some prescribed depth $z = -H$ and thirdly, the condition of Price and Sundermeyer (1999) that the stress (or equivalently within an eddy viscosity framework the current shear) vanishes at this depth, i.e. $\partial U/\partial z = 0$ at $z = -H$. The aim is then to solve this system to investigate the transient motions and the approach to steady-state.

3. Transients in the Ekman layer

The first step towards solving (4) is to apply a Laplace transform

$$U(z, s) = \int_0^\infty e^{-st} U(z, t) \, dt, \quad \text{Re}(s) \geq 0. \quad (6)$$

This converts the partial differential Eq. (4) to an ordinary differential equation

$$\frac{d}{dz} \left[ \nu_e(z) \frac{dU(z, s)}{dz} \right] = ifU(z, s) + sU(z, s) - U(z, 0). \quad (7)$$
The initial condition means the last term is zero. Specific solutions of (7) depend on the
form chosen for the eddy viscosity. Assuming $U(z, t)$ can be established, subject to the
transformed surface stress condition (5) and the appropriate lower boundary condition, the
general solution to (4) is given by the inverse Laplace transform (e.g. Abramowitz and
Stegun, 1972)

$$U(z, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} U(z, s) \, ds, \quad c \geq 0.$$  \hfill (8)

Typically, it is this last step that presents the most difficulties. The mathematical
Appendices A–D describes how contour integration techniques along a particular closed curve, can
be used to evaluate (8). Two specific examples are discussed in detail, the case when $\nu_e = \text{constant (laminar flow)}$ and $\nu_e$ increases linearly with depth, $\nu_e = k u^* z$.

3.1. Constant eddy viscosity layer of infinite depth

When $\nu_e$ is a positive constant, the general solution to (7) is given by

$$U(z, s) = A \exp \left( \frac{(s+i)f}{\nu_e} z \right) + B \exp \left( -\frac{(s+i)f}{\nu_e} z \right).$$  \hfill (9)

The lower boundary condition $U \to 0$ as $z \to -\infty \Rightarrow B = 0$, whilst the Laplace
transformed surface condition

$$\frac{dU(z, s)}{dz} = \frac{\tau_S}{\rho \nu_e s} \text{ at } z = 0,$$  \hfill (10)

$$\Rightarrow A = \frac{\tau_S}{\rho s \sqrt{\nu_e (s+i)}}$$  \hfill (11)

The time-dependent solution is then given by

$$U(z, t) = \frac{\tau_S}{\rho \sqrt{\nu_e}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \exp \left[ \frac{\sqrt{(s+i)f} z}{\nu_e} \right] \frac{1}{(s+i)^{1/2} s} \, ds.$$  \hfill (12)

The contour integral (12) is evaluated in Appendices A–D. The result is

$$U(z, t) = U_C (1 - i) \left[ \exp \left( (1 + i) \frac{z}{d} \right) - \frac{1}{2} \left[ 2 \cosh \left( (1 + i) \frac{z}{\nu_e} \right) \right. \right.$$  

$$- \exp \left( -(1 + i) \frac{z}{\nu_e} \right) \text{erf} \left( \sqrt{\frac{f}{2}} (1 + i) - \frac{z}{2 \sqrt{\nu_e}} \right)$$  

$$- \exp \left( (1 + i) \frac{z}{\nu_e} \right) \text{erf} \left( \sqrt{\frac{f}{2}} (1 + i) + \frac{z}{2 \sqrt{\nu_e}} \right) \right].$$  \hfill (13)

where $d = \sqrt{2\nu_e f}$ is the depth scale of the Ekman layer, $U_C = \tau_S / \rho \sqrt{2\nu_e}$ is the scale
for the surface velocity and $\text{erf}(x + iy)$ represents the complex error function (Abramowitz
Fig. 1. Schematic showing the structure of the unsteady Ekman solution. The viscous terms diffuse momentum a distance $\sqrt{4\nu t}$ into the water column. In region 1, where $z/d > ft$, this flow is unaffected by the Coriolis force and is unidirectional and eastwards. In region 2, where $z/d < ft$, the Coriolis force turns the flow and gives it a southwards component. In region 3, where $ft > 1$, the depth scale of the Ekman layer has reached its steady-state value $d = \sqrt{2\nu/ff}$ and the inertial oscillations are damped on the inertial time scale $ft$.

and Stegun, 1972). This solution was first computed by Fredholm (in a different form) and quoted in Ekman (1905).

This solution goes through three stages of development, as shown schematically in Fig. 1. Firstly, when $ft \ll z/d$, the Coriolis force in the momentum Eq. (4) is negligible compared to the stress gradient, which is therefore solely balances the acceleration (region 1 in Fig. 1), i.e.

$$\frac{\partial U}{\partial t} \sim \nu_e \frac{\partial^2 U}{\partial z^2}. \quad (14)$$

At these short times the error function in (13) can be simplified because $\sqrt{ft/2} \ll z/\sqrt{4\nu_t}$, and the solution becomes the solution for a non-rotating unidirectional current accelerated by a constant stress at the air-sea interface (see Batchelor, 1967, Section 4.4, Eq. (4.4.2)). The depth of this current increases on the viscous scale, namely as $z \sim \sqrt{4\nu_t}$ and flows eastwards at all depths.

Secondly, at longer times $ft \gg z/d$ the flow at shallow depths has been accelerated for sufficiently long that the Coriolis force has had time to deflect the flow towards the steady-state current vector (region 2 of Fig. 1). At these longer times, the error function can again be simplified because $\sqrt{ft/2} \gg z/\sqrt{4\nu_t}$. The solution is then independent of the viscous similarity variable $z/\sqrt{4\nu_t}$, and so can only vary with the inertial variable $ft$. The current suffers damped inertial oscillations about the steady-state Ekman current.

Thirdly, after a time $ft = 1$ momentum has diffused to $z = d$, the depth of the steady-state Ekman current (region 3 in Fig. 1). Hence, for times $ft > 1$, the whole depth of the transient Ekman current is in the final phase of inertial oscillations, which are damped by the viscous stress gradient. Now $\sqrt{ft/2} \gg z/\sqrt{4\nu_t}$ through the whole depth of the Ekman layer and the whole flow is independent of the viscous similarity variable $z/\sqrt{4\nu_t}$. Hence, the inertial oscillations are damped on the inertial time scale $ft$. For very long times, the error function can be approximated (e.g. Press et al., 1992, p. 248) and the current at the surface, $z = 0$,
can be shown to vary as

\[ U(0, t) \rightarrow U_C \left( 1 + \frac{2}{\pi ft} \sin(ft) \right), \tag{15a} \]

\[ V(0, t) \rightarrow U_C \left( -1 + \frac{2}{\pi ft} \cos(ft) \right), \tag{15b} \]

which confirms that the steady-state is achieved only after several inertial oscillations.

This description explains the, perhaps surprising, result that the transients in the Ekman layer do not decay on the viscous time scale, but rather on the inertial time scale. Fig. 2 shows the evolution of the current profile with time. Fig. 2a shows a perspective view of the current profile at times \( ft = 0.1, 1, 4, \infty \). The straight lines that emanate from the vertical solid line show the current vectors at different depths. Fig. 2b shows the evolution of the surface current vector with time \( ft \). At \( ft = 0.1 \), the current is unidirectional; by \( ft = 1 \), the

Fig. 2. (a) Evolution of the normalised current profile, \( U/U_C \), for the unsteady Ekman solution with non-dimensional time \( ft \). Perspective view of the current profile at: \( ft = 0.1 \); dashed line: \( ft = 1 \); dot-dashed line: \( ft = 4 \); solid line: steady-state profile, \( ft \to \infty \).
Fig. 2. (b) Evolution of the normalised surface current from rest, \( U/U_C \) for the unsteady Ekman solution (solid line) and the corresponding solution (16) satisfying a no-slip boundary condition at \( z = -H \) (dashed line), with \( H/d = 1 \). The numbers next to the curves refer to intervals of non-dimensional time \( ft \). The imposition of the no-slip boundary condition means that the solution reaches its steady-state \( \sim 30 \) times faster than the classical Ekman model.

Current is beginning to be turned by the Coriolis force; at \( ft = 4 \), the inertial oscillation has carried the current beyond the steady-state profile denoted by a solid line.

3.2. Ekman layer of finite depth

It is interesting to compare the temporal development of this solution, to those obtained when the influence of the wind is restricted to a finite depth \( z = -H \). Applying the methods discussed here and in the mathematical Appendices A–D, one can show that the solution corresponding to the no-slip condition, \( U = 0 \), applied at \( z = -H \) is
\[ U(z,t) = U_C \left[ \frac{(1 - i) \sinh [(1 + i)z]}{\cosh [(1 + i)d]} \right. \\
\left. + \frac{2de^{-iq}}{H} \sum_{n=0}^{\infty} (-1)^{n+1} \exp[-\alpha_n t][\sin[k_n(H + z)] \right] \right. \\
\left. \cosh[(1 + i)H/d] \right] \left[ (\alpha_n/f)^2 + 1 \right] \right], \tag{16} \]

where \( \alpha_n = (n + 1/2)^2 \pi^2 v_e/H^2 \) and \( k_n = (n + 1/2)\pi/H \). The first term in the square bracket is the steady-state current for Ekman flow in a channel of finite depth and the second term, with the summation, is the transient. At short times, \( ft \ll 1 \), this solution again simplifies to the non-rotating solution (see the related problem discussed in Batchelor, 1967, Section 4.3), with momentum diffused downwards to generate a unidirectional current. (As Batchelor notes, the representation as a series solution given in (16) converges slowly at early times.) At later times, when \( ft > 1 \), the Coriolis force deflects the current southwards. The current vector undergoes inertial oscillations about the steady-state value. The oscillations are in this case damped on the viscous scale, with the mode of wavenumber \( k_n \) decaying on the time scale \( \alpha_n^{-1} \). Hence, the simple mode with \( n = 0 \) persists longest, decaying on the time scale required to diffuse momentum over the depth of the layer, namely \( \alpha_0^{-1} = 4H^2/\pi^2 v_e \).

In this case, the specification of boundary conditions leads to all transients decaying on the viscous time scale. This behaviour is illustrated in Fig. 2b, which shows the evolution of the surface current vector with time \( ft \), in a relatively viscous boundary layer with \( v_e = 0.1 \text{ m}^2 \text{s}^{-1} \) (with \( f = 10^{-4} \text{s}^{-1} \) and \( H = 44.72 \text{ m} \Rightarrow d/H = 1 \)). Note that from this representation of the solution it is not easy to recover (13) for a current in a deep layer (the mathematical limit \( H \gg d \)). The change from transients decaying in the shallow channel on the viscous time scale, to transients decaying in the deep channel on the inertial time scale is therefore not illustrated by this form of the solution.

If instead the stress is fixed at zero at the bottom layer (Price and Sundermeyer, 1999), so that \( \partial U/\partial z = 0 \) at \( z = -H \), the solution for the Ekman current is then

\[ U(z,t) = U_C \left[ \frac{(1 - i) \cosh [(1 + i)z]}{\sinh [(1 + i)d]} \right. \\
\left. + \frac{idH}{H} e^{-iq} \sum_{n=1}^{\infty} (-1)^n \exp[-\beta_n t][\cos[k_n'(z + H)] \right] \left. \sin[k_n(H + z)] \right] \left[ (\beta_n/f)^2 + 1 \right] \right], \tag{17} \]

where \( \beta_n = n^2 \pi^2 v_e/H^2 \) and \( k_n' = n\pi/H \). The important difference here is that there is no mechanism to bring to rest inertial oscillations of modes that induce zero stress at each boundary. The solution (17) shows that the impulsively started wind stress induces a flow \( iU_C d/H \) onto this mode, which indeed continues to oscillate undamped. Physically, this seems somewhat implausible and in the modelling that follows the no-slip boundary condition is applied throughout.

3.3. Eddy viscosity increasing linearly with depth, for an infinitely deep layer

Atmospheric Ekman boundary layers are turbulent and typically one observes a logarithmic velocity profile, corresponding to a linear increase of eddy viscosity with height.
Surface wave velocities, which are often much larger than the mean velocity near the surface, make it much more difficult to establish the presence of a log profile in a turbulent oceanic boundary layer. In addition, turbulence from breaking waves enhances turbulent kinetic energy production and dissipation at the ocean surface, e.g. Terray et al. (1996), and the flow is substantially different to a standard wall layer. However, Richman et al. (1987), Weller and Plueddemann (1996) observed evidence of wind correlated, sub inertial, near surface currents that were strongly sheared, characteristic of a log-layer velocity profile. Additionally, Terray et al. (1999) present observations taken from a series of different datasets, which do exhibit a log profile at depths below one significant wave height. In their large eddy simulations of the ocean mixed layer, McWilliams et al. (1997) computed the eddy viscosity and demonstrated it follows a convex shape, with a roughly linear increase down to the middle of the mixed layer followed by a slow decay. Although a log-layer analytical solution of (4) with this form of viscosity profile is possible, the details are very complicated. So, instead the transient solution with the eddy viscosity increasing linearly with depth throughout the mixed layer is examined. This is a more accurate model than the laminar case, more tractable than the convex shape solution, and should agree with the latter in the upper ocean layers, where the currents are strongest and the experimental observations (see Section 5) most reliable.

Madsen (1977) examined the problem when $v_e = -k\nu_* z$, where $k = 0.4$ is von Karmen’s constant and $u_* = (\tau_S/\rho)^{1/2}$ is the oceanic friction velocity associated with the magnitude of the surface stress. Substituting this form of $v_e$ into (4) and making a convenient change of variables $z_+ = (-z) > 0$ one obtains

$$\frac{\partial U}{\partial t} + i f U - \frac{\partial}{\partial z_+} \left( k\nu_* z_+ \frac{\partial U}{\partial z_+} \right) = 0. \quad (18)$$

Following the method outlined in the preceding section and taking the Laplace transform of (18) gives

$$z_+ \frac{d^2 U}{dz_+^2} + \frac{dU}{dz_+} - \left( \frac{s + if}{k\nu_*} \right) U = 0, \quad (19)$$

assuming $U(z_+, 0) = 0$. The general solution of (19) which satisfies $U(z_+, s) \to 0$ as $z_+ \to \infty$ is

$$U(z_+, s) = BK_0 \left[ 2 \left( \frac{s + if}{k\nu_*} \right)^{1/2} z_+^{1/2} \right], \quad (20)$$

where $K_0[\cdot]$ is a modified Bessel function of the second kind and $B$ is a constant. The stress boundary condition (5) becomes

$$-k\nu_* z_+ \frac{dU}{dz_+} = -k\nu_* z_+ B \left( \frac{s + if}{k\nu_*} \right)^{1/2} \frac{1}{z_+^{1/2}} \times (-1) K_1 \left[ 2 \left( \frac{s + if}{k\nu_*} \right)^{1/2} z_+^{1/2} \right]$$

$$= \frac{\tau_S}{\rho s}, \quad \text{at } z_+ = 0. \quad (21)$$
Using the fact that $K_1(z_+) \xrightarrow{z_+ \to 0} 1/z_+ \Rightarrow B = 2\tau_S/ku_\ast \rho$. Hence, the solution is given by the inverse Laplace transform

$$U(z_+, t) = \frac{2\tau_S}{ku_\ast \rho} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{zt}}{s} K_0 \left[ 2 \left( \frac{s + if}{ku_\ast} \right)^{1/2} z_+^{1/2} \right] ds. \quad (22)$$

This integral is performed in Appendices A–D. The result is

$$U(z_+, t) = \frac{2\tau_S}{ku_\ast \rho} \left[ K_0 \left( \sqrt{2fz_+} \right) (1 + i) - e^{-ift} \int_0^\infty \frac{y}{y^2 + 1} e^{-fy^2} J_0 \left[ 2\sqrt{fy} \sqrt{2fz_+} \right] dy \right]. \quad (23)$$

Without doubt the most accurate way to actually evaluate $U$, for general $z_+$ and $t$ is to calculate the integral in (23) numerically. However, it is not easy to distinguish the analytical behaviour of the time-dependent term as it stands. An equivalent expression to (23) is given by

$$U(z_+, t) = \frac{\tau_S}{ku_\ast \rho} \left[ 2K_0 \left( \sqrt{\frac{2fz_+}{ku_\ast}} (1 + i) \right) + \text{Ei}(-ift) I_0 \left( \sqrt{\frac{2fz_+}{ku_\ast}} (1 + i) \right) \right.$$

$$- e^{-ift} \sum_{n=1}^\infty \frac{(-1)^n}{(n!)^2} \left( \frac{ifz_+}{ku_\ast} \right)^n \sum_{m=1}^n \frac{(m-1)!}{(-if)^m} \left. \right] \quad (24)$$

where $\text{Ei}(\cdot)$ is the exponential integral (Gradshteyn and Ryzhik, 1980, pp. 928–929) and $I_0$ a modified Bessel function of the first kind. For large $ft$ it can be shown that

$$\text{Ei}(-ift) \approx \text{Ci}(ft) + i \left[ \frac{\pi}{2} - \text{Si}(ft) \right] \xrightarrow{ft \to \infty} \frac{1}{ft} \left[ \sin(ft) + i \cos(ft) \right], \quad (25)$$

where $\text{Ci}(ft)$ and $\text{Si}(ft)$ are the cosine and sine integrals (e.g. Press et al., 1992). Notice how the solution varies with time through two distinct dependencies, namely a diffusive dependency $z/\sqrt{\nu t} = \sqrt{z/ku_\ast t}$, and an inertial dependency $ft$. These dependencies parallel the forms of dependency in the solution obtained with a constant eddy viscosity. At early times $ft \ll 1$, the transient varies only through the diffusive dependency, as in the constant viscosity case. In addition, over an inertial period $ft \sim 1$, the diffusion has penetrated a depth $d = ku_\ast / f$. This is the depth scale of the steady-state solution, the first term in Eq. (23). This analysis also parallels the case with constant eddy viscosity. Finally at late times $ft \gg 1$, the second and third terms on the right hand side of (24) vary with time only through the inertial dependence $ft$, as in the constant eddy viscosity solution. However, the transients now decay as $1/ft$ rather than $1/\sqrt{ft}$. This analysis shows how the form of the eddy viscosity does not change the qualitative nature of the three-stage evolution of the transients shown schematically in Fig. 1. In particular, the transients decay on the inertial timescale. This long time for decay of transients places severe tests on observing the steady-state Ekman layer in the real ocean, because the wind stress typically varies on time scales comparable with the inertial time scale.
4. The coupled oceanic–atmospheric Ekman log-layer

4.1. Background

One consequence of the prescribed linear variation $v_e$ of with depth is the logarithmic singularity in the steady-state solution (23) at $z_+ = 0$. Madsen (1977) avoided this problem by introducing a sea surface roughness length scale $z_{0S}$ and actually evaluating the ‘surface’ currents at $z_+ = z_{0S}$. For values of $z_{0S}$ in the range 0.025–0.1 m the typical surface deflection angle from the wind lay in the range 9.1–11.7°, depending upon the windspeed. Hence, a change in the sea surface roughness length by a factor of four, typically only produces a change in the deflection angle of about 15%. This relative insensitivity of the deflection angle to changes in $z_{0S}$ is advantageous, because there is little empirical evidence concerning the behaviour of $z_{0S}$ in the literature. One can also argue that the procedure adopted by Madsen (1977) is flawed mathematically, as the ‘surface’ current and the surface stress boundary condition (5) are no longer consistent, although in practice the physical results are not greatly affected.

Ideally one would like to resolve the problem of the logarithmic singularity in a more satisfactory manner, before proceeding to examine details of the steady-state solution any further. Two modifications suggest themselves. First introducing the sea surface roughness length into the eddy viscosity relation directly, enables one to negate the log singularity at $z_+ = 0$ in a way that is mathematically consistent with the surface stress condition. Second by coupling the oceanic Ekman layer to a corresponding atmospheric Ekman layer, introduces the possibility of linking $z_{0S}$ to observable quantities such as windspeed, wind stress, etc.

4.2. Model formulation

The equations governing the coupled oceanic–atmospheric Ekman log-layer are given by

ocean:
$$\frac{\partial U}{\partial t} + i f U - \frac{\partial}{\partial z_+} \left[ k u_*(z_+ + z_{0S}) \frac{\partial U}{\partial z_+} \right] = 0, \quad z_+ \geq 0,$$

atmosphere:
$$\frac{\partial U}{\partial t} + i f(U - U_G) - \frac{\partial}{\partial z} \left[ k u_*(z + z_{0A}) \frac{\partial U}{\partial z} \right] = 0, \quad z \geq 0.$$

Eq. (26) is simply (18), but with a modified eddy–viscosity relation $v_e(z_+) = k u_*(z_+ + z_{0S})$. This still increases with depth but the introduction of the sea surface roughness length $z_{0S} > 0$ (the subscript ‘S’ denotes oceanic variables, the subscript ‘A’ denotes those that apply in the atmosphere) means that the log-layer solution remains finite as $z_+ \to 0$. In Eq. (27), the eddy viscosity is assumed to increase with height, i.e. $v_e(z_+) = k u_*(z + z_{0A})$ where $z_{0A} > 0$ is a standard atmospheric roughness length, and $U_G$ is the geostrophic wind. The introduction of the roughness lengths $z_{0A}$ and $z_{0S}$ is designed to parameterise the effects of the surface characteristics, most particularly wave motions, on the logarithmic velocity.
profiles. Initially, it will be assumed that $U = 0$ in the ocean for all $z_+$ and $U = U_G$ in the atmosphere for all $z$. Without loss of generality the geostrophic wind will be assumed to lie along the $x$-axis, i.e. $U_G = U_G + i0$.

The lower boundary condition on (26) remains the same, i.e. $U \rightarrow 0$ as $z_+ \rightarrow \infty$, whilst for the atmosphere the upper boundary condition it is assumed that $U \rightarrow U_G$ as $z \rightarrow \infty$. The conditions at the interface are crucial in determining the nature of the solutions obtained. One would expect the wind stress at the surface to be equal to the turbulent stress in the water, i.e. $\tau_S = \rho_A u^2_A \approx \rho_Su^2_{+S}$ (the approximation here reflects that a small proportion of the wind stress is likely to support wave generation and growth). Hence, (5) becomes

$$\tau_S = \rho_A ku_A(z + z_0A) \frac{\partial U}{\partial z} \bigg|_{z_+ \rightarrow 0} = -\rho_S ku_S(z_+ + z_0S) \frac{\partial U}{\partial z} \bigg|_{z_+ \rightarrow 0}. \quad (28)$$

One further condition is needed for a complete solution. The most natural is to assume that the surface current approaches the wind speed at the surface, i.e.

$$U(z_+, t)|_{z_+ \rightarrow 0} = U(z, t)|_{z_+ \rightarrow 0}. \quad (29)$$

### 4.3. Solution

The solution to (28) and (29) subject to the boundary conditions discussed can be found using the method outlined in Section 3 and the mathematical Appendices A–D. The algebra is long and somewhat tedious and won’t be produced here.\(^1\) The final solution is given by

$$U \approx U_G \left[ \frac{\rho_A u_A K_0 \left[ \sqrt{2f(z_+ + z_0S)/ku_A(1 + i)} \right]}{\rho_A u_A K_0 \left[ \sqrt{2fz_0A/ku_A(1 + i)} \right] + \rho_S u_S K_0 \left[ \sqrt{2fz_0A/ku_A(1 + i)} \right]} \right] e^{-\rho_A ku_A z_+} + U_{GE} e^{-\rho_S ku_S z_+} \int_0^\infty \frac{\rho_S u_S K_0 \left[ 2f(z_+ + z_0A)/ku_A(1 + i) \right]}{\rho_A u_A K_0 \left[ \sqrt{2fz_0A/ku_A(1 + i)} \right] + \rho_S u_S K_0 \left[ \sqrt{2fz_0A/ku_A(1 + i)} \right]} dy,$$

$$z_+ \geq 0 \quad (30a)$$

and

$$U \approx U_G - U_G \left[ \frac{\rho_S u_S K_0 \left[ \sqrt{2f(z_+ + z_0A)/ku_A(1 + i)} \right]}{\rho_A u_A K_0 \left[ \sqrt{2fz_0A/ku_A(1 + i)} \right] + \rho_S u_S K_0 \left[ \sqrt{2fz_0A/ku_A(1 + i)} \right]} \right] e^{-\rho_S ku_S z_+} \int_0^\infty \frac{\rho_S u_S K_0 \left[ 2f(z_+ + z_0A)/ku_A(1 + i) \right]}{\rho_A u_A K_0 \left[ \sqrt{2fz_0A/ku_A(1 + i)} \right] + \rho_S u_S K_0 \left[ \sqrt{2fz_0A/ku_A(1 + i)} \right]} dy,$$

$$z \geq 0 \quad (30b)$$

\(^1\) Details can be obtained by writing to the principal author.
where

\[ A(y) = \rho_A u_{sA} J_0 \left[ 2\sqrt{\frac{f_{0A}}{ku_{sA}}} + \rho_S k u_{sA} J_0 \left[ 2\sqrt{\frac{f_{0A}}{ku_{sA}}} \right] \right], \]

\[ B(y) = \rho_A u_{sA} Y_0 \left[ 2\sqrt{\frac{f_{0A}}{ku_{sA}}} + \rho_S k u_{sA} Y_0 \left[ 2\sqrt{\frac{f_{0A}}{ku_{sA}}} \right] \right], \]

where \( J_0, Y_0 \) being Bessel functions and assuming both \( f_{0A}/ku_{sA} \ll 1 \) and \( f_{0S}/ku_{sA} \ll 1 \).

### 4.4. Discussion

The first point to note about (30) is the time-dependent terms have a similar structure to that which appears in the Madsen (1977) solution (23). Hence, it is no surprise that numerical evaluation of (30) reveals that the time-dependent terms are transients, dying off as \( 1/ft \). The characteristics of the steady-state terms are important and are discussed in more detail in the mathematical Appendices A–D. First, one can show that near the surface one obtains a characteristic log-layer solution

\[ U \approx P U_G \log \left( \frac{4 f(z_+ + z_{0S})}{ku_{sA}} \right) \quad \text{provided} \quad \frac{f(z_+ + z_{0S})}{ku_{sA}} \ll 1, \]

where \( P = \rho_A u_{sA} / [\rho_A u_{sA} \log(4f_{0S}/ku_{sA}) + \rho_S u_{sA} \log(4f_{0A}/ku_{sA})] \). Implicit in Eq. (C.1b), given in Appendix C, is the result that provided \( z_{0A} < z_{0S} u_{sA} / u_{sA} = z_{0S} \rho_S / \rho_A \approx 30 z_{0S} \), then the surface current will always be deflected to the right of the geostrophic wind. (In fact, this deflection is an underestimate because normally one observes the deflection from either the surface wind stress or the windspeed at 10 m above the surface \( U_{10} \), quantities which are actually deflected a few degrees to the left of \( U_G \). Further information about the relative size of the roughness lengths can be obtained by looking at the surface stress relationship. From results ((C2) and (C3)) listed in Appendix C (which are good approximations) it makes sense to define \( u_{sA} \) (or \( u_{sA} \)) in terms of the roughness lengths by the following implicit equations

\[ u_{sA}^2 = \frac{-U_G \rho_S u_{sA} k [\rho_A u_{sA} \log(4f_{0S}/ku_{sA}) + \rho_S u_{sA} \log(4f_{0A}/ku_{sA})] \left( \rho_A u_{sA} \log(4f_{0S}/ku_{sA}) + \rho_S u_{sA} \log(4f_{0A}/ku_{sA}) \right)^2 + \pi^2 (\rho_A u_{sA} + \rho_S u_{sA})^2 / 4}{\left( \rho_A u_{sA} \log(4f_{0S}/ku_{sA}) + \rho_S u_{sA} \log(4f_{0A}/ku_{sA}) \right)^2 + \pi^2 (\rho_A u_{sA} + \rho_S u_{sA})^2 / 4} \]

or equivalently

\[ u_{sA}^2 = \frac{-U_G \rho_S u_{sA} k [\rho_A u_{sA} \log(4f_{0S}/ku_{sA}) + \rho_S u_{sA} \log(4f_{0A}/ku_{sA})] \left( \rho_A u_{sA} \log(4f_{0S}/ku_{sA}) + \rho_S u_{sA} \log(4f_{0A}/ku_{sA}) \right)^2 + \pi^2 (\rho_A u_{sA} + \rho_S u_{sA})^2 / 4}{\left( \rho_A u_{sA} \log(4f_{0S}/ku_{sA}) + \rho_S u_{sA} \log(4f_{0A}/ku_{sA}) \right)^2 + \pi^2 (\rho_A u_{sA} + \rho_S u_{sA})^2 / 4} \]

Eq. (32) links a prescribed wind stress and geostrophic wind (or equivalently a measurement of \( U_{10} \)) to the surface roughness lengths. However, to obtain a direct estimate of \( z_{0A} \) and in particular \( z_{0S} \) individually requires further information. One possibility is to use observations of the strength of the current based on the wind. Huang (1979) summarised a series of laboratory and field data measurements, that show the strength of the surface current is
typically 1–4% of the mean wind $U_{10}$. Combining the results ((C1)–(C3)) as outlined in Appendix C gives the following explicit expression for the sea surface roughness length

$$z_{0S} = \frac{k u_* A}{4f} \sqrt{\frac{\rho_A}{\rho_S}} \exp \left[ -q k \sqrt{\frac{\rho_S}{\rho_A}} U_{10} u_* A \right], \quad (33)$$

where $q = 0.01–0.04$ is the relative strength of the current relative to the wind, i.e. $U = q U_{10}$.

Eqs. (32) and (33) provide the means for calculating the roughness length for a given surface stress, windspeed and current strength $q$. This allows one to calculate the steady-state Ekman currents from the general solution (30). Table 1 displays some example results assuming $\tau_S = 0.09 \, N \, m^{-2}$ corresponding (roughly) to an average windspeed $U_{10} = 7 \, m \, s^{-1}$ with $f = 10^{-4} \, s^{-1}$, $\rho_A = 1.2 \, kg \, m^{-1}$ and $\rho_S = 1025 \, kg \, m^{-3}$, assuming a variety of current strengths. (The parameter $U_G$ is adjusted to give the appropriate value of $U_{10} = 7 \, m \, s^{-1}$ at 10 m). Two features are immediately apparent from the results in Table 1. First, relatively low surface current strengths, corresponding to relatively large sea surface roughness lengths, result in large angular deflections. The published observations are inconclusive on this point. Second, the currents beneath the surface are virtually independent of the roughness lengths, as would be expected.

The deflection angles for the stronger currents lie close to the 10$^\circ$ value predicted in Madsen (1977), but generally on the low side when compared to the observations listed in Huang (1979). However, the key question is how close do the characteristics of the predicted, deeper, wind-driven currents reflect reality? To answer this question it is necessary to examine some observational data in more detail.

5. Field observations of upper ocean Ekman layers—modelling implications

Price and Sundermeyer (1999) highlight three distinct observational datasets of Ekman layer currents. The Long-Term Upper Ocean Study (LOTUS3), Briscoe and Weller (1984) and Price et al. (1987) was carried out at 35$^\circ$N during a four month summer period in the western Sargasso Sea. Vector measuring current meters (VMCM) were attached to a tethered surface mooring, from which wind-driven current estimates were derived. This was done by assuming that any wind-driven currents are far more likely to be confirmed primarily to the surface layers, than pressure-driven flows. At a suitable reference depth $H_G$ (50 m in this case, a depth somewhat below the observed mixed layer depth of 10–30 m) the current was assumed to be purely geostrophic, and the wind-driven current above $H_G$ was derived by taking the value of the geostrophic current at the reference depth from the VMCM readings. (The choice of reference depth is somewhat arbitrary, however, Chereskin, 1995, for example, demonstrates that variations of ±10% in $H_G$ do not materially alter the wind current estimates significantly.) The results are summarised in Table 2. (Note that all the current measurements listed in Table 2 are estimates derived from the holographs and three dimensional profiles shown in Fig. 1 of Price and Sundermeyer, 1999.) The average surface wind stress was $\tau_S = 0.07 \, N \, m^{-1}$ (corresponding to an average windspeed $U_{10} \sim 6 \, m \, s^{-1}$) and the daily maximum surface heat flux $Q = 630 \, W \, m^{-2}$. It should be noted that Price and
Table 2
Observed time averaged Ekman current magnitudes and deflection angles for the three experimental datasets discussed in the text

<table>
<thead>
<tr>
<th>Depth z (m) of current ( U )</th>
<th>Current magnitudes and deflection angles from surface stress</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{LOTUS3 dataset}^a )</td>
<td>5 ( 0.046 \text{ m s}^{-1} ) at 77.5° to ( \tau_S )</td>
</tr>
<tr>
<td>10</td>
<td>( 0.028 \text{ m s}^{-1} ) at 94.6° to ( \tau_S )</td>
</tr>
<tr>
<td>15</td>
<td>( 0.019 \text{ m s}^{-1} ) at 96.2° to ( \tau_S )</td>
</tr>
<tr>
<td>25</td>
<td>( 0.006 \text{ m s}^{-1} ) at 140° to ( \tau_S )</td>
</tr>
<tr>
<td>( \text{EBC dataset}^b )</td>
<td>8 ( 0.045 \text{ m s}^{-1} ) at 75.1° to ( \tau_S )</td>
</tr>
<tr>
<td>16</td>
<td>( 0.027 \text{ m s}^{-1} ) at 92.8° to ( \tau_S )</td>
</tr>
<tr>
<td>24</td>
<td>( 0.018 \text{ m s}^{-1} ) at 104° to ( \tau_S )</td>
</tr>
<tr>
<td>32</td>
<td>( 0.011 \text{ m s}^{-1} ) at 121° to ( \tau_S )</td>
</tr>
<tr>
<td>40</td>
<td>( 0.008 \text{ m s}^{-1} ) at 128° to ( \tau_S )</td>
</tr>
<tr>
<td>48</td>
<td>( 0.004 \text{ m s}^{-1} ) at 121° to ( \tau_S )</td>
</tr>
<tr>
<td>( \text{TPHS dataset}^c )</td>
<td>20 ( 0.052 \text{ m s}^{-1} ) at 77.6° to ( \tau_S )</td>
</tr>
<tr>
<td>40</td>
<td>( 0.042 \text{ m s}^{-1} ) at 94.1° to ( \tau_S )</td>
</tr>
<tr>
<td>60</td>
<td>( 0.024 \text{ m s}^{-1} ) at 107° to ( \tau_S )</td>
</tr>
<tr>
<td>80</td>
<td>( 0.007 \text{ m s}^{-1} ) at 103° to ( \tau_S )</td>
</tr>
</tbody>
</table>

These values are estimates, derived from the hodographs and three-dimensional profiles shown in Fig. 1 of Price and Sundermeyer (1999).

\( ^a \tau_S = 0.07 \text{ N m}^{-2}, U_{10} = 6 \text{ m s}^{-1}, f = 8.36 \times 10^{-5} \text{ s}^{-1}, H_G = 50 \text{ m and } Q = 630 \text{ W m}^{-2} \).

\( ^b \tau_S = 0.09 \text{ N m}^{-2}, U_{10} = 7 \text{ m s}^{-1}, f = 8.77 \times 10^{-5} \text{ s}^{-1}, H_G = 60 \text{ m and } Q = 570 \text{ W m}^{-2} \).

\( ^c \tau_S = 0.11 \text{ N m}^{-2}, U_{10} = 8 \text{ m s}^{-1}, f = 2.53 \times 10^{-5} \text{ s}^{-1}, H_G = 150 \text{ m and } Q = 560 \text{ W m}^{-2} \).

**Sundermeyer (1999)** estimate the standard errors in all the current values listed in Table 2 to be roughly 15–20%.

The second set of data (Table 2) is taken from the Eastern Boundary Current observations off the Northern California coast described by Chereskin (1995). Moored acoustic Doppler profiles were used to measure the local currents, with simultaneous windspeed readings, recorded over a four month period from June. Northerly winds predominated, with a mean windspeed of about 7 m s\(^{-1}\) (corresponding to a surface wind stress \( \tau_S \sim 0.09 \text{ N m}^{-2} \)) and \( Q = 570 \text{ W m}^{-2} \). The geostrophic reference depth \( H_G \) was taken to be 60 m (as compared to a maximum mixed layer depth of 40 m) and the corresponding reference current used to calculate the wind-driven currents.

The third set of observations is taken from the transpacific hydrographic survey (TPHS) reported in Wijffels et al. (1994). During a voyage across the Pacific along the line of latitude 10°N, current measurements, temperature profiles and wind readings were recorded. The voyage was divided into three sections and Price and Sundermeyer (1999) averaged data from the first two sections (roughly from the Philippines to 160°W) to produce the wind current profiles summarised in Table 2. The geostrophic reference depth \( H_G \) was set to be 150 m (compared with a mixed layer depth ranging from 50 to 100 m). The winds consistently blew from the north-east at about 8 m s\(^{-1}\) (\( \tau_S \sim 0.11 \text{ N m}^{-2} \)), \( Q = 560 \text{ W m}^{-2} \), and \( f = 2.53 \times 10^{-5} \text{ s}^{-1} \) significantly lower than the other two cases.
One obvious characteristic of all three datasets is the very large deflection angle of the near surface currents, \( \sim 75^\circ \), from the local wind stress. This is much larger than predictions derived from the coupled Ekman log-layer model discussed in the previous section (cf. Tables 1 and 2). A more subtle point, highlighted by Price and Sundermeyer (1999), is that the spirals are ‘flattened’ (i.e. the current decays more rapidly than the current direction rotates to the right) when compared to the classical Ekman spiral derived from (13). To quantify this effect Price and Sundermeyer (1999) introduced a ‘flatness’ parameter \( F \), defined to be the ratio of the amplitude decay to the turning rate

\[
F_z = \frac{dS/dz}{(S(d\theta/dz))},
\]

where \( S \) is the current speed and \( \theta \) the deflection angle (in radians). By definition the classical steady-state Ekman solution has \( F = 1 \) exactly at all \( z \), whilst for the coupled Ekman log-layer solution \( F = 1.3–1.6 \) in the upper layers. By contrast the observed spirals in Table 2 have flatness factors greater than two.

One method to increase the flatness factors in the theoretical solutions is to specify that the wind-driven currents do not penetrate below a certain depth \( H \). However, simply ensuring that \( U \rightarrow 0 \) as \( z_+ \rightarrow H \) in the coupled Ekman log-layer solution given in Section 4, for example, will have the tendency to reduce all the deflection angles listed in Table 1 still further. Hence, this cannot account for the large deflection angles \( \sim 75^\circ \) observed in the near surface currents listed in Table 2. Price and Sundermeyer (1999) argued that variations in density stratification and, in particular, the diurnal cycling of the mixed layer depth (associated with the daytime warming of the upper ocean by solar radiation leading to stable stratification and and a reduction in the mixed layer depth, followed by night time cooling leading to increased vertical mixing and a deepening of the mixed layer) could be responsible. Stratification has the effect of attenuating the Ekman current profiles more rapidly than one would expect to see in a homogeneous ocean layer, manifesting itself in the high flatness factors observed in the datasets. They introduced a two-layer laminar Ekman model, in which the mixed layer depth \( H_{ML}(t) \) was allowed to cycle from a daytime minimum (13, 17 and 25 m for the LOTUS3, EBC and 10°N cases, respectively) to a night time maximum, over a 24 h period (more complex three and four layered models are also discussed to resolve the discontinuities at \( H_{ML}(t) \)). Above \( H_{ML}(t) \), a constant value of \( \nu_e = 1 \text{ m}^2 \text{s}^{-1} \) was used, below \( \nu_e = 0 \). This produced a series of current predictions in line with the observations (cf. Figs. 1 and 11 of their paper).

There are however, a number of problems with the Price and Sundermeyer (1999) approach. First, turbulent boundary layers in the ocean are not laminar as discussed in Section 3. Also of significance is their adoption of a zero stress boundary condition, \( dU/dz_+ = 0 \) at the night time maximum, which is crucial to their results. For a sinusoidal varying \( H_{ML}(t) \) (admittedly the actual variation in \( H_{ML}(t) \) was sawtooth, but this seems unlikely to invalidate the analysis) between 0 and \( 2H \), one can show that the exact steady-state solution is just the steady part of (17). The angular deflections from the wind direction (x-axis) of the steady surface currents given by (16) and (17) are (Ekman, 1905)

\[
\tan(\theta) = \frac{\sinh(2H/d) - \sin(2H/d)}{\sinh(2H/d) + \sin(2H/d)},
\]
and
\[
\tan(\theta) = \frac{\sinh(2H/d) + \sin(2H/d)}{\sinh(2H/d) - \sin(2H/d)},
\]
respectively. For the no-slip boundary condition \( \theta \) in (35) varies from 0° \((H/d = 0)\) to a maximum of 46.57° \((H/d = 1.963)\) before tending to the Ekman value of 45°. When the current shear is zero, \( \theta \) varies in (36) from 90° to a minimum of 43.40° before approaching 45° for large \( H/d \). This latter result illustrates the problem with the zero shear lower boundary condition for the laminar model of Price and Sundermeyer (1999). The predicted surface current deflection angles are too large. For instance using the values adopted by Price and Sundermeyer (1999) for the LOTUS3 data, \( 2H_{ML(ave)} = 50 \text{ m}, v_e = 10^{-1} \text{ m s}^{-1} \) and \( f = 8.77 \times 10^{-5} \text{ s}^{-1} \), implies \( H_{ML(ave)}/d = 0.52 \) and the surface deflection angle \( \theta \approx 80° \), i.e. practically identical to the sub-surface deflection angles listed in Table 2. (This point is also apparent in Fig. 11b of Price and Sundermeyer, 1999, where the predicted surface current deflection for the EBC data \( \sim 67° \) is comparable with the value of 75° observed 8 m below.) Whilst there are no observations of actual surface currents for the three datasets discussed (although Chereskin, 1995 reports that maximum California current surface flows can reach 0.25 \text{ m s}^{-1}), this surface deflection angle is much larger that the typical observed deflection angles lying between about 10 and 45° listed in Huang (1979). On the other hand, adopting the more physically realistic no-slip boundary condition \( U = 0 \) at the night time maximum, would give more appropriate surface current deflections, but not in combination with the large sub-surface deflections actually observed.

The above analysis suggests that a new approach is needed to understand the sub-surface results of Table 2, one that downplays the role of diurnal cycling as against that of surface waves. The justification for this assertion is the relatively large average windspeeds recorded during the observational periods, namely 6–8 \text{ m s}^{-1}. Here relative refers to the suggestion by Leibovich (1983) that a windspeed of 3 \text{ m s}^{-1} is sufficient, whatever the stratification, to set up Langmuir circulations and the subsequent results of Weller and Price (1988) who observed evidence of such motions at even lower windspeeds. Although as yet there is no systematic understanding of all the processes involved, Langmuir circulations lead to enhanced turbulent mixing in the upper ocean layers (via upwelling and downwelling zones associated with adjacent counter rotating Langmuir cells) and a deepening of the mixed layer, e.g. Weller and Price (1988), McWilliams et al. (1997). The effects of the enhanced mixing produced by Langmuir circulations are actually well illustrated by Fig. 8 of Price and Sundermeyer (1999), which shows a weekly period taken from the LOTUS3 data. During the first two days when the wind stress was relatively small \(<0.07 \text{ N m}^{-2}\), large sea temperature differentials in the upper 10 m of the ocean were recorded, combined with a distinct diurnal variation of \( H_{ML} \). On the remaining five days during stronger winds, the sea temperature differentials were suppressed and the diurnal variation is much less apparent. Hence, the observations suggest evidence of a strong vertical mixing mechanism, compatible with the presence of Langmuir circulations. Consequently, a homogeneous model of the upper ocean mixed layer extending down to the ‘permanent’ (minus the diurnal fluctuating component due to solar heating) mixed layer depth, but one that includes the effects of surface waves, seems more likely to explain the observed currents in Table 2. For in addition to producing Langmuir circulations and hence changing the mixing within the mixed layer, the Stokes
drift associated with the surface waves also act upon the planetary vorticity and thereby change the dynamics of the whole mean flow. The next section investigates this effect quantitatively.

6. The coupled oceanic–atmospheric ‘Stokes–Ekman’ layer

6.1. Model formulation

In their comprehensive review of the subject, Xu and Bowen (1994) present an Eulerian model of wind and wave-driven flow confirmed to a finite depth. They deduce that for constant viscosity flows the presence of water waves modifies the standard Ekman equations (cf. Eqs. (3)–(5)) by the addition of three terms. First, the wave induces a virtual tangential stress on the free surface \( \tau \sim 0.03–0.12 \tau_S \), primarily along the direction of wave propagation (Longuet-Higgins, 1953). Secondly, a Coriolis-induced wave stress (directly proportional to the Stokes drift) is set up in the interior of the flow, directed at right angles to the direction of wave propagation. Finally, a wave stress is induced by friction at a solid lower boundary, parallel to the direction of wave propagation. Of these, the latter is significant only within a layer of order \( O(\sqrt{2\nu/\sigma}) \) (where \( \sigma \) is the wave frequency) and is ignored here, whilst the wave stress at the surface will be absorbed as part of the total wind and wave stress at the surface \( (\tau = \tau_S + \tau_W \text{ in Eq. (5)}) \). Hence, the horizontal momentum ‘Stokes–Ekman’ equations (see also Leibovich, 1977b; McWilliams et al., 1997) take the form

\[
\frac{\partial U}{\partial t} + i f(U + U_S) = \frac{\partial}{\partial z} \left( \nu(z) \frac{\partial U}{\partial z} \right), 
\]

(37)

where the term if \( U_S \) represents deformation of the planetary vorticity \( f \) by the Stokes drift, \( U_S \). The latter is estimated by assuming that the ocean surface consists of steady, monochromatic deep-water waves of the form

\[
\eta = a \cos(k \cdot x - \sigma t),
\]

(38)

where \( \eta(x, y, t) \) is the position of the free surface \( (z = 0 \text{ the average surface height}) \), \( a \) the wave amplitude, \( k = (k_x, k_y) \) the wavenumber and \( \sigma = (g|k|)^{1/2} \text{ the wave frequency.} \)

The Stokes drift associated with such a wave is

\[
U_S(z, t) = U_Se^{2|k|z} \hat{s}(t),
\]

(39)

where \( U_S = \sigma/k|a^2 \) (Philips, 1977) and \( \hat{s}(t) \) is a unit vector in the \( x-y \) plane. When the eddy viscosity varies linearly with depth the coupled oceanic–atmospheric Ekman layer equations incorporating a Stokes drift term are given by

\[
\frac{\partial U}{\partial t} + i fU - \frac{\partial}{\partial z_+} \left( k\nu_S(z_+ + z_{0S}) \frac{\partial U}{\partial z_+} \right) = -ifU_Se^{-2|k|z+} \hat{s}(t), \quad z_+ \geq 0,
\]

(40)

\[
\frac{\partial U}{\partial t} + i f(U - U_G) - \frac{\partial}{\partial z} \left( k\nu_A(z + z_{0A}) \frac{\partial U}{\partial z} \right) = 0, \quad z \geq 0.
\]

(41)

Eqs. (40) and (41) must be solved subject to the boundary and coupling conditions. As in Section 4, \( U \to U_G \text{ as } z \to \infty \text{ and the coupling conditions are that the flow speed
and shear stress are continuous at the surface \( z = 0 \). The observations subtract from the measured current the geostrophic current at \( z = H_G \) and hence in effect the observations of the wind-driven current have \( U \to 0 \) as \( z_+ \to H_G \). Hence, this boundary condition is applied to the model. In addition, the direction of the Stokes drift \( \hat{s} \) and hence the wave field must be specified. The surface wave field is generated by the surface stress (e.g. Belcher and Hunt, 1998) and so \( \hat{s}(t) \) lies parallel to the instantaneous direction of \( \partial U / \partial z_+ \) at \( z_+ = 0 \). This imposes a constraint on the solution enabling one to calculate \( \hat{s}(t) \).

6.2. Steady-state solution

The discussion here is restricted to the steady solution to (40) and (41). However, in passing it should be noted that the nature of the time-dependent terms is unaffected by the Stokes drift. Over a semi-infinite domain they remain transients, dying off as \( 1/\sqrt{ft} \), in a similar manner to the time-dependent terms appearing in the standard oceanic–atmospheric Ekman log-layer solution (31) discussed earlier.

The equivalent steady-state equations of (40) and (41) are

\[
\frac{d^2 U}{dz_+^2} + \frac{1}{(z_+ + z_{0S})} \frac{dU}{dz_+} - \frac{ifU}{ku_S(z_+ + z_{0S})} = \frac{iyU_S e^{-2k|z_+}}{ku_S(z_+ + z_{0S})} \hat{s} \quad (42)
\]

\[
\frac{d^2 U}{dz^2} + \frac{1}{(z + z_{0A})} \frac{dU}{dz} - \frac{ifU}{ku_A(z + z_{0A})} = 0 \quad (43)
\]

where \( \hat{s} \) is now a constant vector. The general solution to (42) and (43) is given by

\[
U = U_G + R_1 K_0 \left[ 2 \left( \frac{if}{ku_A} \right)^{1/2} (z + z_{0A})^{1/2} \right], \quad (44)
\]

\[
U = R_2 I_0 \left[ 2 \left( \frac{if}{ku_A} \right)^{1/2} (z_+ + z_{0S})^{1/2} \right]
+ R_3 K_0 \left[ 2 \left( \frac{if}{ku_A} \right)^{1/2} (z_+ + z_{0S})^{1/2} \right] + \text{PI}(z_+), \quad (45)
\]

where the particular integral \( \text{PI}(z_+) \) is given by

\[
\text{PI}(z_+) = \frac{2ifU_S \hat{s}}{ku_A} \left[ -I_0 \left[ 2 \sqrt{\frac{if(z_+ + z_{0S})}{ku_A}} \right] \int_{z_+}^{H_G} K_0 \left[ 2 \sqrt{\frac{if(z_+ + z_{0S})}{ku_A}} \right] e^{-2k|z_+|} dz_+ \right]
+ K_0 \left[ 2 \sqrt{\frac{if(z_+ + z_{0S})}{ku_A}} \right] \int_{z_+}^{H_G} I_0 \left[ 2 \sqrt{\frac{if(z_+ + z_{0S})}{ku_A}} \right] e^{-2k|z_+|} dz_+ \right]. \quad (46)
\]

In (44), the condition \( U = U_G \) as \( z \to \infty \) has been imposed, whilst \( \text{PI}(z_+ \to H_G) = 0 \) in (46). The three remaining constants \( R_1, R_2 \) and \( R_3 \) must be chosen to satisfy \( U(z_+ \to H_G) = \)
the changes are relatively small, only of the order of 8%. Hence, for simplicity, Eq. (33) is retained with no modification. 

Employing the standard assumptions concerning the roughness lengths that both \( f_{0A} / k u_A \ll 1 \) and \( f_{0S} / k u_{sA} \ll 1 \) one can show (after considerable algebraic manipulation) that

\[
U(z) = U_0 - \left[ \frac{U_G \rho_A k^2 u_A u_{sA} - 2 f_{0A} h_0}{k u_{sA}} f_0 [f(z')] e^{-3k c' dz'} + T \int_0^{H_0} f_0 [f(z')] e^{-3k c' dz'} \right] \\
\times \frac{p_s}{\rho_A} \left( h_0 [f(H_G)] + K_0 \int_0^{H_0} f_0 [f(z')] e^{-3k c' dz'} \right) \\
+ \int_0^{H_G} h_0 [f(z')] e^{-3k c' dz'} \right] \right) \right] k_0 \left[ \frac{f(z + z_0A)}{k u_A} \right] \\
(47)
\]

and

\[
U(z_+) = \left[ \frac{U_G \rho_A k^2 u_A u_{sA} - 2 f_{0A} h_0}{k u_{sA}} f_0 [f(z')] e^{-3k c' dz'} + T \int_0^{H_0} f_0 [f(z')] e^{-3k c' dz'} \right] \\
\times \left[ h_0 [f(H_G)] + K_0 \left[ \frac{f(z_+ + z_0S)}{k u_{sA}} \right] - K_0 \int_0^{H_0} f_0 [f(z')] e^{-3k c' dz'} \right] \\
(48)
\]

where

\[
T = k \left[ u_{sA} \rho_s K_0 \left[ \frac{f(z_+ + z_0S)}{k u_{sA}} \right] \right] + u_{sA} \rho_A K_0 \left[ \frac{f(z_+ + z_0A)}{k u_{sA}} \right] \\
\]

\[
Q = 2 f_{0S} \rho_s K_0 \left[ \frac{f(z_+ + z_0S)}{k u_{sA}} \right] - \rho_A k u_{sA} I_0 \left[ \frac{f(z_+ + z_0A)}{k u_{sA}} \right] \\
\]

and

\[
f(x) = \frac{i f(x + z_0S)}{k u_{sA}}. \\
\] (49c)

As yet the constant \( \hat{s} \) is undetermined. The mathematical Appendices A–D illustrates how \( \hat{s} \) can be calculated from (48) so that it lies in the direction of the surface stress \( -dU/dz_+ \) at \( z_+ = 0 \). In addition, it derives an equivalent expression to (32), which can be used to calculate \( z_0A \) provided \( z_0S \) is known. The presence of the Stokes drift term also leads to a correction in Eq. (33) used to prescribe \( z_0S \) in terms of the strength of the surface current relative to the wind. It turns out that to first order the effect of the waves is to reduce \( z_0S \). This results in an increase in the magnitude of the surface current and a reduction in the deflection angle relative to the surface stress. However, for the datasets discussed here, the changes are relatively small, only of the order of 8%. Hence, for simplicity, Eq. (33) is retained with no modification.
All that remains is to specify the values of $U_S$ and $|k|$ the parameters governing the Stokes drift (39), for the three experimental campaigns discussed in Section 5. Unfortunately, direct observations of sea state, significant wave heights, etc. were not recorded during the trials, so inevitably there is a speculative element in the choice of values. Komen et al. (1994) has published a series of wave growth equations, which summarise the observations from a number of independent experimental trials. Two different scenarios will be investigated here. First, it is assumed that the wave growth was fetch-limited, in which case Komen et al. (1994) gives two non-dimensional expressions (Eqs. (2.226c) and (2.226d)), based on composite dataset of all the observations made) for the wave amplitude $a$ and peak frequency $\sigma$, namely

$$\frac{g^2a^2}{16u_*^4A} = 6.5 \times 10^{-4} \left( \frac{g\Gamma}{u_*^2A} \right)^{0.9},$$

$$\frac{\sigma u_*A}{g} = 3.08 \left( \frac{g\Gamma}{u_*^2A} \right)^{-0.27},$$

where $\Gamma$ is the fetch (distance from windward shore). The actual fetch values are not known and so the value of $\Gamma$ was adjusted to give the best fit between the observed Ekman currents (Table 2) and the predicted values. Second it is assumed that the waves were fully developed, in which case the following relations for $a$ and $\sigma$ can be employed (cf. Eqs. (6.71a) and (6.71b) of Komen et al., 1994)

$$\frac{g^2a^2}{16u_*^4A} = 1.1 \times 10^3, \quad \frac{\sigma u_*A}{g} = 2\pi \times 5.6 \times 10^{-3}.$$  

(52)

From these relations, it is simple matter to calculate the corresponding values of $U_S$ and $|k|$. 

### 6.3. Discussion

Table 3 presents a comparison of the model results obtained by optimising $\Gamma$ with the observed Ekman current values. (The vector misfit $\varepsilon = N^{-1}\sum_{i=1}^{N}(U_M - U_O)^2$ of Price and Sundermeyer, 1999, where $U_M$ is the model current and $U_O$ the observed current at the $N$ different depths, was employed as a ‘goodness of fit’ criterion for the selection of $\Gamma$.) For illustrative purposes, the surface current was assumed to be 3% of the wind, i.e. $q = 0.03$ in (33). (The choice of $q$ has very little influence on the sub-surface currents as shown in Table 1). As one can see this produces model surface currents in the 10–45° range typically observed, in combination with much larger sub-surface deflections in the range 72–80°, as seen in the three experimental datasets discussed. This is a direct result of the introduction of the Stokes drift term (39). This point is further illustrated by the corresponding hodographs of Fig. 3a–c, which show the coupled log-layer solution (48) both with and without the Stokes drift term. Incorporating this term clearly improves the agreement between the model and the experimental measurements. There is also good general agreement at the lower levels.
developed sea:

For instance, the vector misfit and percentage variance (PV), where

\[ PV = 100 \left( 1 - \varepsilon \left[ \frac{1}{N} \sum_{i=1}^{N} \left| U_{oi} \right|^2 \right]^{-1} \right), \]  

(53)

The parameters \( \Gamma \) (for the fetch model) and \( q \) are fixed, the other parameters can either be derived from them or are known a priori.

\[ \varepsilon = 0.009 \text{N m}^{-2}, \quad H_G = 50 \text{ m}, \quad q = 0.03, \quad u_{AA} = 3.6 \times 10^{-3} \text{ m s}^{-1}, \quad u_{AA} = 0.24 \text{ m s}^{-1}, \quad u_{AA} = 8.3 \times 10^{-3} \text{ m s}^{-1}. \]

The parameters \( \Gamma \) (for the fetch model) and \( q \) are fixed, the other parameters can either be derived from them or

\[ \varepsilon = 0.011 \text{N m}^{-2}, \quad H_G = 60 \text{ m}, \quad q = 0.03, \quad u_{AA} = 5.3 \times 10^{-3} \text{ m s}^{-1}, \quad u_{AA} = 1.6 \times 10^{-3} \text{ m s}^{-1}, \quad u_{AA} = 0.27 \text{ m s}^{-1}, \quad u_{AA} = 9.4 \times 10^{-3} \text{ m s}^{-1}. \]

The parameters \( \Gamma \) (for the fetch model) and \( q \) are fixed, the other parameters can either be derived from them or

\[ \varepsilon = 0.012 \text{N m}^{-2}, \quad H_G = 150 \text{ m}, \quad q = 0.03, \quad u_{AA} = 3.6 \times 10^{-3} \text{ m s}^{-1}, \quad u_{AA} = 3.9 \times 10^{-3} \text{ m s}^{-1}, \quad u_{AA} = 0.30 \text{ m s}^{-1}, \quad u_{AA} = 1.0 \times 10^{-2} \text{ m s}^{-1}. \]

The parameters \( \Gamma \) (for the fetch model) and \( q \) are fixed, the other parameters can either be derived from them or

\[ \varepsilon = 0.11 \text{N m}^{-2}, \quad H_G = 150 \text{ m}, \quad q = 0.03, \quad u_{AA} = 3.6 \times 10^{-3} \text{ m s}^{-1}, \quad u_{AA} = 3.9 \times 10^{-3} \text{ m s}^{-1}, \quad u_{AA} = 0.30 \text{ m s}^{-1}, \quad u_{AA} = 1.0 \times 10^{-2} \text{ m s}^{-1}. \]

The parameters \( \Gamma \) (for the fetch model) and \( q \) are fixed, the other parameters can either be derived from them or

\[ \varepsilon = 0.010 \text{N m}^{-2}, \quad H_G = 98 \%, \quad u_{AA} = 0.018 \text{ m s}^{-1}, \quad u_{AA} = 5.7 \times 10^{-3} \text{ m s}^{-1}, \quad u_{AA} = 3.00 \text{ m}, \quad u_{AA} = 0.46 \text{ m s}^{-1}, \quad u_{AA} = 3.2 \times 10^{-4} \text{ m s}^{-1}. \]

The parameters \( \Gamma \) (for the fetch model) and \( q \) are fixed, the other parameters can either be derived from them or

\[ \varepsilon = 0.010 \text{N m}^{-2}, \quad H_G = 98 \%, \quad u_{AA} = 0.018 \text{ m s}^{-1}, \quad u_{AA} = 5.7 \times 10^{-3} \text{ m s}^{-1}, \quad u_{AA} = 3.00 \text{ m}, \quad u_{AA} = 0.46 \text{ m s}^{-1}, \quad u_{AA} = 3.2 \times 10^{-4} \text{ m s}^{-1}. \]

The parameters \( \Gamma \) (for the fetch model) and \( q \) are fixed, the other parameters can either be derived from them or

\[ \varepsilon = 0.010 \text{N m}^{-2}, \quad H_G = 98 \%, \quad u_{AA} = 0.018 \text{ m s}^{-1}, \quad u_{AA} = 5.7 \times 10^{-3} \text{ m s}^{-1}, \quad u_{AA} = 3.00 \text{ m}, \quad u_{AA} = 0.46 \text{ m s}^{-1}, \quad u_{AA} = 3.2 \times 10^{-4} \text{ m s}^{-1}. \]
‘goodness of fit’ criteria match the best values obtained by Price and Sundermeyer (1999), and the flatness parameters are greater than two, as appropriate. The associated predictions of average wave amplitudes in the 0.8–2.9 m range seem reasonable. The optimum fetch values for the EBC and LOTUS3 datasets are 51 and 58 km, respectively. The model Ekman currents are actually fairly insensitive to changes in the fetch value. A 50% increase in the fetch of the EBC data for example, only produces an 11% increase in the magnitude of the deflection current at 8 m and increases the deflection angle by less than 9%. The other datasets produce similar results. Notice too that for the LOTUS3 and EBC datasets, the fully developed wave model (Table 3, column 3) gives comparable estimates for current magnitudes and deflection angles, and similar values for the vector misfit and PV to the optimised fetch-limited case. These results support the general robustness of the modelling methodology.

![Figure 3](image-url)

**Fig. 3.** Hodographs of the lower half of the coupled oceanic–atmospheric ‘Stokes–Ekman’ log-layer solution based on the parameters corresponding to the three experimental datasets (a) LOTUS3, (b) EBC and (c) TPHS discussed in the text. The curve denoted by ++, is found simply by setting $U_z = 0$ in (46) and (48). The observed points are derived from the results in Table 3 (left column). The solid curve shows the fetch-limited ‘Ekman–Stokes’ solution, also summarised in Table 3 (middle column).
For the TPHS data the optimum fetch is substantially larger, as one might expect for a cruise across the central Pacific. This too shows good agreement with the observations. On this occasion, the fully developed wave model gives rather different results to the fetch-limited case, the former predicting a Stokes drift term that is too small to produce deflection angles in the range 70–80° near the surface. More observations of waves and Ekman currents are the only way to resolve this. The greatest discrepancies for all the datasets occur at the lowest levels, e.g. $U(25)$ for the LOTUS3 data and $U(80)$ for the TPHS data, where there is a tendency to overestimate the current magnitude and underestimate the deflection angle. This is not surprising because it is at these sort of depths that the effects of density stratification will become apparent, leading to a breakdown of the assumed linear relation between eddy viscosity and depth. The 'permanent' mixed layer depth is about 30 m in the LOTUS3 case and between 50 and 100 m for the TPHS data. Below these levels turbulence, mixing, and eddy diffusivity will be greatly reduced compared to the mixed layer. This will lead to a more rapid attenuation of the Ekman currents than is predicted in the model results.
7. Conclusions

This paper sets out to try to answer two specific questions. First, what is the nature of the unsteady terms contributing to Ekman currents? Second, how can one reconcile with theory the three observed features of the Ekman current, namely surface current deflections of between $10^\circ$ and $45^\circ$ from the wind stress, the much larger sub-surface current deflections $\sim 75^\circ$ and the rapid current attenuation with depth?

With reference to the first question it is worth emphasising the difficulties one faces when trying to formulate any sort of response. Real Ekman currents are the product of a host of interrelated factors, including wind stress, surface wave motion and surface heating, which vary over space and time. Hence the temporal variation of the Ekman currents will be of enormous complexity. Resolving all the contributions that make up the currents is far beyond the scope of even the most powerful and sophisticated computer models. Instead, employing
idealised and simplified models of the Ekman layer yields insights into the unsteady motion that can arise.

To this end a general analytical solution method has been developed here which provides a means for attaining just such insights. The transients in an unsteady Ekman layer are well illustrated by the canonical case of an impulsively started wind stress over a deep ocean, with turbulent mixing parameterised with a constant eddy viscosity. The evolution of the Ekman current follows three stages. Firstly, at early times, the Coriolis acceleration is small and a unidirectional current develops, deepening through diffusive momentum transfer. Secondly, the Coriolis acceleration deflects the near surface current southwards. Thirdly, once the diffusive momentum transport has mixed the current to its steady-state depth, the transients evolve and decay, but on the inertial time scale. Hence, the steady-state is reached after several inertial oscillations. This knowledge of the exact nature of the transient terms, such as those in (16) and (17) is also useful when generating computer simulations of oceanic Ekman layers. Typically, one needs to know how long to ‘spin up’ the solution from rest, to ensure that all the transient terms in the solution have died off.

The key to answering the second question raised here is to appreciate that for moderate to strong windspeeds, the influence of the wave motion via the Stokes drift is fundamental to understanding the observed Ekman current profiles. The model developed here combines a Stokes drift term with a turbulent closure model in which the eddy viscosity varies linearly with depth, which is then substituted into the standard homogeneous Ekman boundary layer equations. The use of a homogeneous upper ocean boundary layer, as opposed to a density stratified one, is justified because there is evidence of enhanced vertical mixing, probably caused by Langmuir circulations which would be induced by the Stokes drift. The resulting solutions show all three characteristics of the observed Ekman layer, in particular the model exhibits good agreement with the measured sub-surface current magnitudes and directions, whilst at the same time making plausible predictions at the surface. This combination results in a velocity profile which is highly sheared in the upper few metres of the ocean mixed layer, a conclusion at odds with for instance the model Price and Sundermeyer (1999), but supported by the observations of Churchill and Csanady (1983) and Weller and Plueddemann (1996). The assumption of Price and Sundermeyer (1999) that the uppermost ocean layers are ‘slab like’ with comparatively little vertical shear, seems erroneous, resulting as it does in surface current deflections close to 75°, outside the observed 10–45° range. Stokes drift will also influence transport in the Ekman layer and may have implications for the representation of the Coriolis-Stokes term in general circulation models (Polten et al., 2004). However, these questions can only be definitively resolved by future experimental observations of near surface currents. The present work re-enforces the view that surface waves change qualitatively the nature of the Ekman layer and so argue forcefully that future observational studies should include some measure of sea surface state, at least the significant wave height for estimating the Stokes drift.

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Appendix A. Integration of Eq. (12)

The first step in performing integrals of this type is to make the substitution $w = (s + if)^{1/2}$, $\Rightarrow dw = (s + if)^{-1/2}/2 ds$. This transforms Eq. (12) to

$$U(z, t) = \frac{\tau \rho}{\sqrt{\nu}} \frac{2}{2\pi i} e^{-it} e^{\frac{1}{\nu} \int_{\sqrt{c+if}}^{\sqrt{c-if}} \frac{e^{w^2/\nu}wz/\sqrt{\nu}}{(w^2 - if)} \, dw.}$$  \hspace{1cm} (A.1)$$

The benefit of making this substitution is that it removes the awkward $\sqrt{s+if}$ term which frequently appears in the integrand, at the expense of complicating the limits of integration. One now needs to perform the integration along a contour defined by $(c + ix)^{1/2}$ where $x \in [-M, M]$ say and then subsequently let $M \to \infty$. Such a contour would look like the portion of the curve labelled $C_1$ in Fig. A.1. One obvious way to perform this contour

Fig. A.1. Argand diagram for $z = x + iy$, $\text{Re} x \ge 0$, showing a plot of the closed curve $C = C_1 + C_2 + C_3 + C_4$, around which the contour integration described in Appendices A–D is performed. Here, $c$ and $M$ are positive real numbers, where $c$ is fixed and subsequently $M$ will be assumed to $\to \infty$. For the integrals discussed in Appendices A–D, the integrands have a simple pole at $z = (1+1)i\sqrt{\nu}$ roughly as shown.
integral would be to extend $C_1$ in such a way to form a closed curve. In that case provided the integrand is analytic, except perhaps at a finite number of simple poles, the integral around $C$ (in an anticlockwise sense) is equal to zero, or the sum of the residues at the poles. Just such a closed curve is illustrated in Fig. A.1. It consists of the contour $C_3$, from $2^{-1/2}[\sqrt{c^2 + M^2} + c]^{1/2}$ to $-2^{-1/2}[\sqrt{c^2 + M^2} + c]^{1/2}$ (here the positive square root is assumed unless otherwise stated) and two curves labelled $C_2$ and $C_4$ joining $C_1$ and $C_3$ together. The curves $C_2$ and $C_4$ are defined in the following way

$$C_2(\theta) = \frac{1}{\sqrt{2}}[\sqrt{c^2 + M^2} + c]^{1/2} + R_2(\theta)e^{i\theta},$$

(A.2a)

where

$$R_2(\theta) = \frac{4}{\pi} \left[ \left[ c + \sqrt{c^2 + M^2} \right]^{1/2} - \frac{1}{\sqrt{2}}(\sqrt{c^2 + M^2} - c) \right] \theta + \frac{3}{\sqrt{2}}[\sqrt{c^2 + M^2} - c]^{1/2} - 2[\sqrt{c^2 + M^2} + c]^{1/2}, \quad \theta \in \left[ \frac{\pi}{2}, \frac{3\pi}{4} \right].$$

(A.2b)

and $C_4(\theta)$ is the complex conjugate of $C_2(\theta)$. (The advantage of this choice $C_2$ is that if $w \in C_2(\theta)$ then $\text{Re} \ w^z \ll 0$ for large $M$, except when $\theta \in [\pi/2, \pi/2 + c/M]$. This means that the ubiquitous $e^{w^z}$ term in the integrand such as (A.1), will ensure the integral along $C_2$ is small for a wide class of problems.) For this particular case,

$$\left| \int_{C_2} \frac{e^{w^z}e^{uz/\sqrt{\nu e}}}{(w^2 - 1)} \, dw \right| \leq Q \int_{\pi/2}^{3\pi/4} \left| \frac{dR_2}{d\theta} + iR_2(\theta) \right| |e^{i\theta}| \, d\theta,$$

(A.3)

and

$$\left| \frac{e^{w^z}e^{uz/\sqrt{\nu e}}}{(w^2 - 1)} \right| \leq Q \quad \text{for} \ w \in C_2(\theta).$$

(A.4)

The integral in (A.3) just gives the length of the contour. For large $M$ this is given approximately by

$$\int_{\pi/2}^{3\pi/4} \left| \frac{dR_2}{d\theta} + iR_2(\theta) \right| |e^{i\theta}| \, d\theta \approx M^{1/2} \int_{\pi/2}^{3\pi/4} \left[ \left( \frac{3}{2} - \sqrt{2} \right) \left( \frac{16}{\pi^2} + \theta^2 \right) + \frac{8\theta}{\pi} \left( \frac{5}{\sqrt{2}} - \frac{11}{4} \right) + \left( \frac{17}{2} - 3\sqrt{2} \right) \right]^{1/2} \, d\theta \approx 2.3M^{1/2}.$$  

(A.5)

To obtain an estimate for $Q$ one should note that when $w \in C_2(\theta)$, $|e^{w^z}| \leq e^{c^z}$ (equality when $\theta = \pi/2$), $|e^{uz/\sqrt{\nu e}}| \leq 1$ (equality when $\theta = 3\pi/4$, assuming $z/\sqrt{\nu e} < 0$ and $|w^2 - 1|^{-1} < 2/M$ for all $\theta \in [\pi/2, 3\pi/4]$), and hence

$$\left| \frac{e^{w^z}e^{uz/\sqrt{\nu e}}}{(w^2 - 1)} \right| < \frac{2e^{c^z}}{M} = Q.$$

(A.6)
This means that
\[ \left| \int_{C_2} \frac{e^{u^2} e^{wz/j\sqrt{\nu_c}}}{(u^2 - i f)} \, dw \right| < \frac{2e^{ci}}{M} \times 2.3M^{1/2} = \frac{4.6e^{ci}}{M^{1/2}} \xrightarrow{M \to \infty} 0, \]  
(A.7)

and so the integrals along \( C_2 \) and \( C_4 \) make no contribution to the total contour integration around \( C \). The function \( e^{u^2} e^{wz/j\sqrt{\nu_c}}/(u^2 - i f) \) has a simple pole at \( w = \sqrt{f/2}(1 + i) \) and so the contour integral around \( C \) is given by
\[
\int_{\sqrt{f/2} - i\infty}^{\sqrt{f/2} + i\infty} \frac{e^{u^2} e^{wz/j\sqrt{\nu_c}}}{w^2 - i f} \, dw + \int_{i\infty}^{-i\infty} \frac{e^{wz/j\sqrt{\nu_c}}}{y^2 - 1} \, dy \\
= 2\pi \text{Res} \left[ w = \sqrt{f/2}(1 + i) \right] = 2\pi i \frac{e^{(f/2)z/j\sqrt{\nu_c}(1+i)}}{2\sqrt{f/2}(1 + i)}. \tag{A.8}
\]

Making the substitution \( y = -ix \) in the second integral on the LHS of (A.8) and rearranging gives
\[
\frac{1}{2\pi i} \int_{\sqrt{f/2} - i\infty}^{\sqrt{f/2} + i\infty} \frac{e^{u^2} e^{wz/j\sqrt{\nu_c}}}{w^2 - i f} \, dw \\
= \frac{(1 - i)e^{(f/2)z/j\sqrt{\nu_c}(1+i)}}{2\sqrt{f/2}} - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2} e^{-iz/j\sqrt{\nu_c}} \, dx \\
= \frac{(1 - i)e^{(f/2)z/j\sqrt{\nu_c}(1+i)}}{2\sqrt{f/2}} - \frac{2}{2\pi} \int_{0}^{\infty} e^{-x^2} \cos \left[ \frac{zx}{\sqrt{\nu_c}} \right] \, dx. \tag{A.9}
\]

The last result is because the integrand is an even function of \( x \). Gradshteyn and Ryzhik (1980, p. 497, No. 3.954) show that it is given by
\[
\int_{0}^{\infty} e^{-x^2} \cos \left[ \frac{zx}{\sqrt{\nu_c}} \right] \, dx \\
= \left[ \frac{\pi e^{i\theta}}{4 \sqrt{f/2}(1 + i)} \right] \left\{ 2\cosh \left( \frac{f}{2\sqrt{\nu_c}}(1 + i)z \right) - e^{-\sqrt{f/2\nu_c}(1+i)z} \text{erf} \left( \frac{\sqrt{f/2}(1 + i)}{2\sqrt{\nu_c}} - \frac{z}{2\sqrt{\nu_c}} \right) - e^{\sqrt{f/2\nu_c}(1+i)z} \text{erf} \left( \frac{\sqrt{f/2}(1 + i)}{2\sqrt{\nu_c}} + \frac{z}{2\sqrt{\nu_c}} \right) \right\}. \tag{A.10}
\]

Substituting results (A.10) and (A.9) into (A.1) gives the final expression (13) quoted in the main part of the paper. To obtain results (16) and (17) the contour illustrated in Fig. A.1 has to be modified to exclude a series of simple poles that occur at points \( \pm i(n + 1/2)\pi \sqrt{\nu_c}/H\sqrt{f} \) or \( \pm i(n\pi \sqrt{\nu_c}/H\sqrt{f}, n = 0, 1, 2, \ldots \), respectively, on the imaginary axis. This can be done
by drawing a small semi-circle around each pole. Standard complex analysis methods (e.g. Churchill and Brown, 1984) enable one to perform the required integration whilst at the same time as allowing the radius of the semi-circles to tend to zero, giving rise to the summation terms in (16) and (17).

**Appendix B. Integration of Eq. (22)**

The integration is performed much as described in Appendix A. First making the substitution \( w = (s + if)^{1/2} \) transforms the integral to

\[
U(z_{+}, t) = \frac{4\tau e^{-i\theta}}{\rho ku_{s}2\pi i} \int_{\sqrt{c+i\infty}}^{\sqrt{c-i\infty}} \frac{we^{w^2t}}{w^2 - if} K_0[2w\sqrt{z_{+}/ku_s}] \, dw. \tag{B.1}
\]

Let \( f(\xi) = \xi e^{x^2/2} K_0[2\xi \sqrt{z_{+}/ku_s}] / (\xi^2 - if) \), where \( \xi \) is any complex number. The integral (B.1) can be found by considering the contour integral of around the closed curve \( C \) illustrated in Fig. A.1. One has

\[
\int_{C} f(\xi) \, d\xi = \int_{\sqrt{c+iM}}^{\sqrt{c-iM}} \frac{we^{w^2t}}{w^2 - if} K_0[2w\sqrt{z_{+}/ku_s}] \, dw + \int_{C_2} f(\xi) \, d\xi = \int_{\sqrt{c-iM}}^{\sqrt{c+iM}} \frac{ye^{y^2t}}{(y^2 - if)} K_0[2y\sqrt{z_{+}/ku_s}] \, dy + \int_{C_4} f(\xi) \, d\xi = 2\pi i \sum \text{residues at poles in } C. \tag{B.2}
\]

Taking the limit as \( M \to \infty \) the integrals along the portions of the curve labelled \( C_2 \) and \( C_4 \) can be shown to tend to zero. The integrand \( f(\xi) \) has one simple pole at \( \xi = \sqrt{if/2}(1 + i) \) enclosed by \( C \). Making the substitution \( y = -ix \) in the third integral of (B.2) and rearranging gives

\[
\int_{\sqrt{c+i\infty}}^{\sqrt{c-i\infty}} \frac{we^{w^2t}}{w^2 - if} K_0[2w\sqrt{z_{+}/ku_s}] \, dw = 2\pi i \text{Res} \left[ \frac{\xi e^{-x^2/2}}{2(1 + i)} \right] - \int_{-\infty}^{\infty} \frac{xe^{-x^2/2}}{x^2 + if} K_0[-2ix\sqrt{z_{+}/ku_s}] \, dx. \tag{B.3}
\]

The complex modified Bessel function of the second kind can be written in the form (Gradshteyn and Ryzhik, 1980)

\[
K_0[-ix] = -\text{Log} \left( -\frac{ix}{2} \right) J_0[x] + \sum_{m=0}^{\infty} \frac{(-x^2)^m \Psi(m + 1)}{2^{2m}(m!)^2}, \tag{B.4}
\]

for any real \( x \), where \( J_0[x] \) is the zeroth order regular Bessel function and Euler’s psi function (Gradshteyn and Ryzhik, 1980, p. 943) is defined to be \( \Psi(1) = -\gamma \), \( \Psi(m + 1) = 1 + 1/2 + 1/3 + \cdots + 1/m - \gamma \) for \( m \geq 1 \), and Euler’s constant \( \gamma = 0.577215 \ldots \). The
logarithmic term in (B.4) is defined as

\[
\log \left( \frac{-ix}{2} \right) = \begin{cases} 
\log \left( \frac{x}{2} \right) - i \pi \frac{x}{2}, & x > 0 \\
\log \left( \frac{-x}{2} \right) + i \pi \frac{x}{2}, & x < 0
\end{cases}
\]  

(B.5)

Substituting (B.4) and (B.5) into the last integral in (B.3) gives

\[
\int_{-\infty}^{\infty} x e^{-x^2} t e^{\frac{-i f}{2} \sqrt{2} \sqrt{z} + \frac{ku}{ku}} \, dx = \int_{-\infty}^{\infty} x e^{-x^2} t e^{\frac{-i f}{2} \sqrt{2} \sqrt{z} + \frac{ku}{ku}} \, dx
\]

(B.6)

The integrand in the last integral is an odd function of \( x \) and hence the integral must be zero. Likewise the two integrals involving the log terms are identical but of opposite sign, and cancel. Hence, (B.6) reduces to

\[
\int_{-\infty}^{\infty} x e^{-x^2} t e^{\frac{-i f}{2} \sqrt{2} \sqrt{z} + \frac{ku}{ku}} \, dx = i \pi \frac{1}{2} \int_{-\infty}^{\infty} x e^{-x^2} t e^{\frac{-i f}{2} \sqrt{2} \sqrt{z} + \frac{ku}{ku}} \, dx
\]

(B.7)

The residue term in (B.3) is given by

\[
\text{Res} \left[ \zeta = \frac{f}{2} (1 + i) \right] = \frac{\zeta e^{-\zeta} t}{\zeta + \sqrt{2} (1 + i)} K_0 \left[ 2 \zeta \sqrt{\frac{z}{ku}} \right] \bigg|_{\zeta = \sqrt{2} (1 + i)} = \frac{e^{\sqrt{2} (1 + i)}}{2} K_0 \left[ 2 \zeta \sqrt{\frac{z}{ku}} - (1 + i) \right].
\]

(B.8)
The latter integral is obtained by making the substitution \( u \), hence, substituting results (B.7) and (B.8) into (B.3) the total solution (B.1) is given by

\[
U(z, t) = \frac{4\pi e^{-\psi}}{\rho ku^2} \left[ \pi i e^{i\psi} K_0 \left[ 2\sqrt{\frac{2e^{z^2}}{ku_*^2}} (1 + i) \right] - i\pi \int_0^\infty \frac{xe^{-x^2t}}{x^2 + if} J_0 \left[ 2\sqrt{\frac{z^2}{ku_*}} \right] dx \right]
\]

\[
= \frac{2\pi}{\rho ku_*} \left[ K_0 \left[ 2\sqrt{\frac{2e^{z^2}}{ku_*^2}} (1 + i) \right] - e^{-\psi} \int_0^\infty \frac{xe^{-x^2t}}{x^2 + if} J_0 \left[ 2\sqrt{\frac{z^2}{ku_*}} \right] dx \right],
\]

which is (23) before making the substitution \( \psi = \sqrt{\pi} \). To examine the analytical behaviour of the time dependent term it is necessary to expand \( J_0 \) as a series, giving

\[
e^{-\psi} \int_0^\infty \frac{xe^{-x^2t}}{x^2 + if} J_0 \left[ 2\sqrt{\frac{z^2}{ku_*}} \right] dx
\]

\[
= e^{-\psi} \sum_{n=0}^{\infty} \frac{(-1)^n (z^2 + ku_*)^n}{(n!)^2} \int_0^\infty \frac{x^{2n+1} e^{-x^2t}}{x^2 + if} dx
\]

\[
= \frac{e^{-\psi}}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (z^2 + ku_*)^n}{(n!)^2} \int_0^\infty u^n e^{-u^2t} du.
\]

The latter integral is obtained by making the substitution \( u = x^2 \). Its evaluation can be found listed in Gradshteyn and Ryzhik (1980, p. 312, No. 3.3535), giving

\[
\frac{e^{-\psi}}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (z^2 + ku_*)^n}{(n!)^2} \left[ (-1)^{n-1} (i f)^n e^\psi \text{Ei}(-i f t) + \sum_{m=1}^{n} (m - 1)! (-i f)^{n-m} t^m \right]
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n-1} (i f)^n (2z^2 + ku_*^2)^n}{(n!)^2} \text{Ei}(-i ft) + \frac{e^{-\psi}}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (ifz^2 + ku_*^2)^n}{(n!)^2} \sum_{m=1}^{n} (m - 1)! (-i f)^m t^m
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} \frac{\left[ (1 + i) \sqrt{2z^2 + ku_*^2}/4 \right]^n}{(n!)^2} \text{Ei}(-i ft) \]

\[
+ \frac{e^{-\psi}}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (ifz^2 + ku_*^2)^n}{(n!)^2} \sum_{m=1}^{n} (m - 1)! (-i f)^m t^m
\]

\[
= \frac{1}{2} I_0 \left[ (1 + i) \sqrt{\frac{2z^2}{ku_*}} \right] \text{Ei}(-i ft) + \frac{e^{-\psi}}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (ifz^2 + ku_*^2)^n}{(n!)^2} \sum_{m=1}^{n} (m - 1)! (-i f)^m t^m,
\]

which is the result quoted in (24). However, in practice the summation terms prove difficult to evaluate accurately and it is more reliable to integrate (B.9) numerically.
Appendix C. Characteristics of the steady-state coupled oceanic–atmospheric
Ekman log-layer solution, Eq. (30)

Provided \( f(z_+ + z_{OS})/ku_S \ll 1 \) in Eq. (30) one can make use of the approximation
\( K_0[\zeta] = -\log \zeta = -\log |\zeta| - \arg(\zeta) \) for \(|\zeta| \ll 1\), and the steady-state current close to the
surface is given approximately by

\[
U \approx \frac{U_G \rho A u_A \log[4f(z_+ + z_{OS})/ku_S]}{[(e_A V_A \log[4f_{20A}/ku_S] + \rho S u_S \log[4f_{20A}/ku_A])^2 + \pi^2(\rho A u_A + \rho S u_S)^2/4]}
\]

(C.1a)

\[
U \to_{z_+} \frac{U_G \rho A u_A \log[4f_{20A}/ku_A]}{[(e_A V_A \log[4f_{20A}/ku_S] + \rho S u_S \log[4f_{20A}/ku_A])^2 + \pi^2(\rho A u_A + \rho S u_S)^2/4]}
\]

(C.1b)

For small \( z_+ \) the real part of (C.1a) dominates the imaginary part and one obtains a character-
istic log-layer solution near the surface given by (31). The steady-state surface stress of
(30) is given by

\[
-k u_S (z_+ + z_{OS}) \frac{dU}{dz_+} \approx -\frac{U_G \rho A u_A u_S k [\rho A u_A \log[4f_{20S}/ku_S] + \rho S u_S \log[4f_{20A}/ku_A]}
{-i(\pi/2)[\rho A u_A + \rho S u_S]}

+ (\pi^2(\rho A u_A + \rho S u_S)^2/4)^
\]

(C.2)

using the result \( dK_0[\zeta]/d\zeta = -K_1[\zeta] \) and \( K_1[\zeta] \approx 1/\zeta \) if \(|\zeta| \ll 1\). The log terms in (C.2)
mean that the magnitude of the real part will dominate the magnitude of the imaginary part
(typically by a factor of about 8, as the surface stress is deflected to the left of \( U_G \) by about
6–8°). Hence, to good approximation one can define

\[
u^2_S = \frac{-U_G \rho A u_S u_A k [\rho A u_A \log[4f_{20S}/ku_S] + \rho S u_S \log[4f_{20A}/ku_A]]}
{[\rho A u_A \log[4f_{20S}/ku_S] + \rho S u_S \log[4f_{20A}/ku_A])^2 + \pi^2(\rho A u_A + \rho S u_S)^2/4]}
\]

(C.3)

into which is Eq. (32a). Eq. (32b) is obtained by substituting \( u^2_S = u^2_A \rho A / \rho S \) left hand
side of (C.3).

To obtain an explicit expression for the sea surface roughness length \( z_{OS} \) in terms of
the magnitude of the surface current relative to the windspeed consider Eqs. (32b) and (C.1b).
Assuming the real part of (C.1b) is dominated by the log terms in the numerator, dividing
through by (32b) gives
\[
\frac{U}{u^*_A} \approx -\frac{\rho_A \log[4f_0S/ku_{as}]}{\rho_S ku_{as}} \Rightarrow \frac{U}{u^*_A} = -\sqrt{\frac{\rho_A \log[4f_0S/\rho_S \rho_A/ku_{as}]}{ku_{as}}}. \tag{C.4}
\]
Hence, assuming that \( U = qU_0 \), where \( q = 0.01-0.04 \), simple algebraic manipulation gives
\[
z_{0S} = \frac{ku_{as}}{4f} \sqrt{\frac{\rho_A}{\rho_S}} \exp \left[ -qk \sqrt{\frac{\rho_S}{\rho_A} U_0} \right], \tag{C.5}
\]
which is Eq. (33).

Appendix D. Determination of the direction of the Stokes drift \( \hat{s} \)

One would expect that the wave field is generated by the surface wind stress and hence the direction of the Stokes drift \( \hat{s} \) lies in the direction \(-\partial U/\partial z_+|_{z_+ \to 0}\) (see (28)). So, the first task is to evaluate \(-\partial U/\partial z_+|_{z_+ \to 0}\) from (48). After some algebraic manipulation this can be shown to be
\[
-\frac{dU}{dz_+|_{z_+ \to 0}} \approx [ku_{as}[T_0[f(H_G)] + QK_0[f(H_G)]]^{-1}
\]
\[
\times \left\{ U_G \rho_A k^2 u_{as} S \left[ \frac{1}{2z_{0S}} I_0[f(H_G)] + \left( i f/k u_{as} S \right) K_0[f(H_G)] \right] - i U_S S \left[ I_0[f(H_G)] \int_0^{H_G} K_0[f(z')] e^{-2k|z'|} dz' - K_0[f(H_G)] \right] \times \int_0^{H_G} I_0[f(z')] e^{-2k|z'|} dz' \right\} \left[ \frac{Q}{z_{0S}} - \frac{2iT}{ku_{as}} \right], \tag{D.1}
\]
where \( T, Q \) and \( f(x) \) are given by Eqs. (49a)–(49c), respectively. If the direction of \( \hat{s} \) and \(-\partial U/\partial z_+|_{z_+ \to 0}\) coincide, one can rewrite (D.1) in the very much simpler form
\[
|W|(x + iy) = (A + iB) + (C + iD)(x + iy), \tag{D.2}
\]
where
\[
A + iB = \frac{U_G \rho_A k^2 u_{as} S [(1/2z_{0S}) I_0[f(H_G)] + (i f/ku_{as}S) K_0[f(H_G)]]}{ku_{as}[T_0[f(H_G)] + QK_0[f(H_G)]]}, \tag{D.3a}
\]
\[
C + iD = \frac{-i U_S S [I_0[f(H_G)] \int_0^{H_G} K_0[f(z')] e^{-2k|z'|} dz'] \left( Q/\sqrt{z_{0S}} \right)}{ku_{as}[T_0[f(H_G)] + QK_0[f(H_G)]]}, \tag{D.3b}
\]
\[
\hat{s} = x + iy, \quad x^2 + y^2 = 1, \tag{D.3c}
\]
\[ |W| = \left| - \frac{dU}{dz} \right|_{z_+ \to 0} . \] (D.3d)

One would like to obtain expressions for the unknown real values \( x \), \( y \) and \( |W| \) in terms of \( A \), \( B \), \( C \) and \( D \). From (D.2) one has

\[ |W|x = A + Cx - yD \]
\[ |W|y = B + Cy - Dx \]  
\Rightarrow \]
\[ x = \frac{(|W| - C)A - BD}{(|W| - C)^2 + D^2} \]  
\[ y = \frac{AD + (|W| - C)B}{(|W| - C)^2 + D^2} \]  
whilst the constraint \( x^2 + y^2 = 1 \) \Rightarrow \]
\[ |W| = C \pm \sqrt{A^2 + B^2 - D^2}. \] (D.6)

In practice, the positive square root is needed to ensure \( |W| \geq 0 \). Hence, (D.5) becomes

\[ x = \frac{\sqrt{A^2 + B^2 - D^2}}{A^2 + B^2} A - BD \]
\[ y = \frac{AD + \sqrt{A^2 + B^2 - D^2}}{A^2 + B^2} B \]  
Eq. (D.7) gives expressions that allow for the evaluation of \( \tilde{s} \). The only problem is that \( A \), \( B \), \( C \) and \( D \) depend on the surface roughness lengths \( z_{0A} \) and \( z_{0S} \) which are currently unknown. However, it is the supposition of this paper that the sea surface roughness can be specified in terms of the strength of the surface current relative to the wind, and Eq. (33) holds (ignoring the small correction due to the Stokes drift for the reasons given after (49)). Hence, all that is required is \( z_{0S} \) and this can be found from the surface stress boundary condition (28)

\[ u^2_{sS} = -ku_{sS}(z_+ + z_{0S}) \left| \frac{dU}{dz} \right|_{z_+ \to 0} . \] (D.8)

Now assuming both \( f_{z0A}/ku_{sA} \ll 1 \) and \( f_{z0S}/ku_{sS} \ll 1 \), Eq. (D.2) is dominated by the real terms and (D.8) becomes

\[ u^2_{sS} \approx ku_{sS}z_{0S}|W|x. \] (D.9)

Using (D.6) and (D.7) in (D.9) gives

\[ \frac{z_{0S} \left[ C + \sqrt{A^2 + B^2 - D^2} \right] \left[ A\sqrt{A^2 + B^2 - D^2} - BD \right]}{A^2 + B^2} - \frac{u_{sS}}{k} = 0. \] (D.10)

This implicit constraint, equivalent to Eq. (32) in the coupled oceanic–atmospheric Ekman log-layer, provides a means of estimating \( z_{0A} \) from a known \( z_{0S} \). Solving (D.10) allows one
to calculate $\hat{\mathbf{s}}$ from Eq. (D.7). Typically, if $U_G$ is chosen to lie along the $x$ (real) axis (as is the case for all the results here) $\hat{\mathbf{s}}$ is directed some 6–8° to the left of $U_G$ (e.g. see the expressions for $r_S$ listed in Table 3).

References


