A Mathematical Example Displaying Features of Turbulence

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Introduction

Before entering upon the study of the example in question we want to make some introductory remarks about the actual hydrodynamic problems, in particular, about what is known and what is conjectured concerning the future behavior of the solutions. Consider an incompressible and homogeneous viscous fluid within given material boundaries under given exterior forces. The boundary conditions and the outside forces are assumed to be stationary, i.e. independent of time. For that, it is not necessary that the walls be at rest themselves. Parts of the material walls may move in a stationary movement provided that the geometrical boundary as a whole stays at rest. An instance is a fluid between two concentric cylinders rotating with prescribed constant velocities or a fluid between two parallel planes which are translated within themselves with given constant velocities. As to the stationarity of the exterior forces we may cite the case of a flow through an infinitely long pipe with a pressure drop (regarded as an outside force). In this case the pressure drop is required to be a given constant independent of time.

Each motion of the fluid that is theoretically possible under these conditions satisfies the Navier-Stokes equations ($\rho = 1$)

\[ \frac{\partial u_i}{\partial t} = - \sum_j u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \mu \Delta u_i \]  

(0.1)

together with the incompressibility condition

\[ \sum \frac{\partial u_i}{\partial x_i} = 0 \]

(0.2)

and the given stationary boundary conditions. To an arbitrarily prescribed initial velocity field $u(x, 0)$ satisfying (0.2) and the boundary conditions there is expected to belong a unique solution $u(x, t; \mu)$ ($t \geq 0$) of (0.1) and (0.2) that fulfills these boundary conditions. The pressure $p(x, t; \mu)$ may be considered as an auxiliary variable which, at every moment $t$ is (up to an additive constant) perfectly well determined by the instantaneous velocity field $u(x, t)$ (solution of a Neumann problem of potential theory). If $p$ is eliminated in this manner the Navier-Stokes equations appear in the form of an integrodifferential space-time system for the $u_i$ alone where the right hand sides consist of first and second degree terms in the $u_i$. 

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It is convenient to visualize the solutions in the phase space $\Omega$ of the problem. A phase or state of the fluid is a vector field $u(x)$ in the fluid space that satisfies (0.2) and the boundary conditions. The totality $\Omega$ of these phases is therefore a functional space with infinitely many dimensions. A flow of the fluid represents a point motion in $\Omega$ and the totality of these phase motions forms a stationary flow in the phase space $\Omega$, which, of course, is to be distinguished from the fluid flow itself. What is the asymptotic future behavior of the solutions, how does the phase flow behave for $t \to \infty$? And how does this behavior change as $\mu$ decreases more and more? How do the solutions which represent the observed turbulent motions fit into the phase picture? The great mathematical difficulties of these important problems are well known and at present the way to a successful attack on them seems hopelessly barred. There is no doubt, however, that many characteristic features of the hydrodynamical phase flow occur in a much larger class of similar problems governed by nonlinear space-time systems. In order to gain insight into the nature of hydrodynamical phase flows we are, at present, forced to find and to treat simplified examples within that class. The study of such models has been originated by J. M. Burgers in a well known memoir.\textsuperscript{1} His principal example is essentially

\[
\begin{align*}
\frac{\partial v}{\partial t} &= -v \frac{\partial v}{\partial x} + w \frac{\partial w}{\partial x} + v - w + \mu \frac{\partial^2 v}{\partial x^2} \\
\frac{\partial w}{\partial t} &= w \frac{\partial v}{\partial x} + v \frac{\partial w}{\partial x} + v + w + \mu \frac{\partial^2 w}{\partial x^2}
\end{align*}
\]

where $0 \leq x \leq 1$ and where the boundary conditions are $v = w = 0$ at $x = 0$ and $x = 1$. Though simpler in form than the hydrodynamic equations this example presents essentially the same difficulties and the future behavior of the solutions for small values of $\mu$ still is an unsolved problem.

In this paper another nonlinear example is presented and studied that differs from Burgers' model in that the future behavior of its solutions can be completely determined. In this respect our example seems to us to be the first of its kind. The detailed study of this space-time system reveals geometrical features of the phase flow which come close to the qualitative picture we believe to prevail in the hydrodynamic cases. It must, however, be said that, for reasons to appear later in the paper, the analogy does not extend to the quantitative relations found to hold in turbulent fluid flow.

The observational facts about hydrodynamic flow reduced to the case of fixed side conditions and with $\mu$ as the only variable parameter are essentially these: For $\mu$ sufficiently large, $\mu > \mu_0$, the only flow observed in the long run is a stationary one (laminar flow). This flow is stable against arbitrary initial

\textsuperscript{1}J. M. Burgers, \textit{Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion}. Akademie van Wetenschappen, Amsterdam, Eerste Sectie, Deel XVII, No. 2, pp. 1–53, 1939.
disturbances. Theoretically, the corresponding exact solution is known to exist for every value of \( \mu > 0 \) and its stability in the large can be rigorously proved, though only for sufficiently large values of \( \mu \). The corresponding phase flow in phase space \( \Omega \) thus possesses an extremely simple structure. The laminar solution represents a single point in \( \Omega \) invariant under the phase flow. For \( \mu > \mu_0 \), every phase motion tends, as \( t \to \infty \), toward this laminar point. For sufficiently small values of \( \mu \), however, the laminar solution is never observed. The turbulent flow observed instead displays a complicated pattern of apparently irregularly moving "eddies" of varying sizes. The view widely held at present is that, for \( \mu > 0 \) having a fixed value, there is a "smallest size" of eddies present in the fluid depending on \( \mu \) and tending to zero as \( \mu \to 0 \). Thus, macroscopically, the flow has the appearance of an intricate chance movement whereas, if observed with sufficient magnifying power, the regularity of the flow would never be doubted.

The qualitative mathematical picture which the author conjectures to correspond to the known facts about hydrodynamic flow is this: To the flows observed in the long run after the influence of the initial conditions has died down there correspond certain solutions of the Navier-Stokes equations. These solutions constitute a certain manifold \( \mathcal{M} = \mathcal{M}(\mu) \) in phase space invariant under the phase flow. Presumably owing to viscosity \( \mathcal{M} \) has a finite number \( N = N(\mu) \) of dimensions. This effect of viscosity is most evident in the simplest case of \( \mu \) sufficiently large. In this case \( \mathcal{M} \) is simply a single point, \( N = 0 \). Also the complete stability of \( \mathcal{M} \) is in this simplest case obviously due to viscosity. On the other hand, for smaller and smaller values of \( \mu \), the increasing chance character of the observed flow suggests that \( N(\mu) \to \infty \) monotonically as \( \mu \to 0 \). This can happen only if at certain "critical" values

\[
\mu_0 > \mu_1 > \mu_2 > \cdots \to 0
\]

the number \( N(\mu) \) jumps. The manifold \( \mathcal{M}(\mu) \) itself presumably changes analytically as long as no critical value is passed. Now we believe that when \( \mu \) decreases through such a value \( \mu_k \) a continuous branching phenomenon occurs. The manifold \( \mathcal{M}(\mu) \) of motions observed in the long run (more precisely its analytical continuation for \( \mu < \mu_k \)) loses its stability. The notion of stability here refers to the whole manifold and not to the single motions contained in it. The loss of stability implies that the motions on the analytically continued \( \mathcal{M} \) are no longer observed. What we observe after passing \( \mu_k \) is not the analytical continuation of the previous \( \mathcal{M} \) but a new manifold \( \mathcal{M}(\mu) \) continuously branching away from \( \mathcal{M}(\mu_k) \) and slightly swelling in a new dimension. This new \( \mathcal{M}(\mu) \) takes over stability from the old one. Stability here means that the "majority" of phase motions tends for \( t \to \infty \) toward \( \mathcal{M}(\mu) \). We must expect that there is a "minority" of exceptional motions that do not converge toward \( \mathcal{M} \) (for instance the motions on the analytical continuation of the old \( \mathcal{M} \) and of all the other manifolds left over from all the previous branchings). The simplest case of such a bifurcation with corresponding change of stability...
is the branching of a periodic motion from a stationary one. This case is clearly observed in the flow around an obstacle (transition from the laminar flow to a periodic one with periodic discharges of eddies from the boundary). The next simplest case is the branching of a one-parameter family of almost periodic solutions from a periodic one. The new solutions are expressed by functions

\[ u(\phi_1, \phi_2; \mu) \]

periodic in each \( \phi \) with period \( 2\pi \) where

\[ \phi_i = a_i t + c_i, \quad a_i = a_i(\mu), \]

and where the \( c_i \) are arbitrary constants (we can without loss of generality assume \( c_i = 0 \)). The functions \( f \) with \( \phi \), arbitrary describe the manifold \( M(\mu) \) which, in our case, is of the type of a torus. If \( M \), quite generally, continuously develops out of the laminar point there is a reasonable expectation that \( M \) is a multidimensional torus-manifold described by functions

\[ u(\phi_1, \cdots, \phi_N; \mu) \]

with period \( 2\pi \) in each of the \( \phi \) and that the turbulent solutions are given by linear functions \( \phi_i = a_i(\mu)t + c_i \) as before. This is what happens in our example which precisely exhibits this phenomenon of continuous growth of almost periodic solutions out of the laminar one with an infinite succession of branchings of the type described above.

The geometrical picture of the phase flow is, however, not the most important problem of the theory of turbulence. Of greater importance is the determination of the probability distributions associated with the phase flow, particularly of their asymptotic limiting forms for small \( \mu \). In the case of our example these distributions have limiting forms (normal distribution). Recent investigations, however, suggest that there are essential deviations from normality in the hydrodynamic case. It seems that the influence of the second degree terms is in this case essentially different and much more complicated than in the case of our over-simplified model.

Another observation on our model case is this: If we proceed to the limit \( \mu \to 0 \) within the "observed," i.e. the turbulent solutions the turbulent fluctuations are found to disappear and we obtain a special stationary solution in the "ideal case" (equations with \( \mu = 0 \)). This shows, by way of analogy, how important a role viscosity plays in turbulence.

**Formulation of the Problem**

The space of our model is a one-dimensional circular line and our space variable is an angular variable \( x \mod 2\pi \). All space functions are thus periodic functions of \( x \) with period \( 2\pi \). For two arbitrary space functions \( f, g \) we denote
by

\[ f \circ g = \frac{1}{2\pi} \int_0^{2\pi} f(x + y)g(y) \, dy \]

their convolution product which is again a space function. \( f \circ 1 \) is a constant, the mean value of \( f(x) \) over a period. Throughout this paper \( z^* \) denotes the conjugate of the complex number \( z \). Our integrodifferential system written in complex form is

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -z \circ z^* - u \circ 1 + \mu (\partial^2 u / \partial x^2), \\
\frac{\partial z}{\partial t} &= z \circ u^* + z \circ F^* + \mu (\partial^2 z / \partial x^2),
\end{align*}
\]

where \( \mu > 0 \) is a parameter and where

\[
F(x) = a(x) + ib(x)
\]

is an arbitrarily given complex-valued space function. \( F(x) \) is supposed to be an even and absolutely integrable function of \( x \),

\[
F(-x) = F(x).
\]

Further conditions upon \( F \) will be stated when they are needed. The unknowns are the two complex-valued functions \( u(z, t) \) and \( z(z, t) \). The real equations into which (1.0) splits up are four in number.

In what follows we confine ourselves to those solutions of (1.0) for which \( u, z \) are even functions of \( x \) and for which \( u \) is real. It will be proved, by using (1.2), that any solution \( u, z \) which is even for \( t = 0 \) must be even for all \( t > 0 \) and that, for such a solution, \( u \) is always real if it is real for \( t = 0 \). If we confine ourselves to the even solutions with \( u \) real, (1.0) splits upon setting

\[
z = v + iw
\]

into three real equations for \( u, v, w \)

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -v \circ v - w \circ w - u \circ 1 + \mu (\partial^2 u / \partial x^2), \\
\frac{\partial v}{\partial t} &= v \circ u + v \circ a + w \circ b + \mu (\partial^2 v / \partial x^2), \\
\frac{\partial w}{\partial t} &= w \circ u - v \circ b + w \circ a + \mu (\partial^2 w / \partial x^2),
\end{align*}
\]

where \( F(x) = a(x) + ib(x) \). Our problem is to study the real solutions of (2.0) which are even functions of \( x \) with period \( 2\pi \).

Another equivalent but in some respects more straightforward formulation of our problem is obtained if we confine ourselves to the interval

\[ 0 \leq x \leq \pi. \]

We look for the real solutions of (2.0), where

\[
\begin{align*}
f \circ g &= \frac{1}{2\pi} \int_0^\pi f(\vert x - y \vert)g(y) \, dy + \frac{1}{2\pi} \int_0^\pi f(\pi - \vert x - y \vert)g(\pi - y) \, dy.
\end{align*}
\]
satisfying the boundary conditions

\[(3.0) \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = \pi.\]

The equivalence of this formulation is a consequence of the following facts. An even function of period \(2\pi\) may be arbitrarily prescribed in the interval \((0, \pi)\). If the first derivative exists for all \(x\), \((3.0)\) must be satisfied. On the other hand, if a function in \([0, \pi]\) has a second derivative in this closed interval and if \((3.0)\) is satisfied the corresponding even and periodic function has a second derivative at every \(x\) (in particular, the first derivative is continuous throughout). That the convolution of two even space functions reduces, for \(0 \leq x \leq \pi\), to the expression mentioned above follows from a simple calculation.

In what follows we use the handier complex form \((1.0)\) of the problem with restriction to the even solutions whether \(u\) is real or not. Our second degree terms share an important property with those in the hydrodynamic case. When the time derivative of the kinetic energy

\[
\frac{1}{2} \int_{0}^{2\pi} (uu^* + zz^*) \, dx, \quad \frac{1}{2} \int_{0}^{2\pi} (u^2 + v^2 + w^2) \, dx
\]

is computed from the equations the third degree terms obtained on the right hand side are found to cancel. In our case this follows from the identity

\[
\int_{0}^{2\pi} (f \circ g) h \, dx = \int_{0}^{2\pi} (f \circ h) g \, dx.
\]

Our second degree terms, however, strikingly differ in nature from the hydrodynamic ones in that they are pure integrals. The fact that they are convolutions enables one to calculate the solutions by spatial Fourier analysis.

**Properties of the Even Solutions of \((1.0)\) and \((2.0)\)**

There is an infinite number of critical values \(\mu_1 > \mu_2 > \mu_3 > \cdots \to 0\) for \(\mu\) with the following properties. For \(\mu > \mu_1\) there is a stationary "laminar solution" which is stable in the large for \(t \to +\infty\), i.e. which will, as \(t \to +\infty\), be approached by any other solution for the same value of \(\mu\). For \(\mu_n > \mu > \mu_{n+1}\) there is an \(n\)-dimensional manifold of "turbulent solutions" essentially stable in the large as \(t \to +\infty\), i.e. "almost" all other solutions will approach some of the turbulent solutions with the same value of \(\mu\). These turbulent solutions are represented by almost periodic functions of \(t\) of very simple type. These solutions persist for \(\mu < \mu_{n+1}\) but they are no longer stable. For any given value of \(\mu\) there is a definite statistical distribution within the totality of the "velocity fields," i.e. in the function space of the sets of three arbitrary functions \(u(x), v(x), w(x)\) satisfying the given boundary conditions. These statistics are simply defined by time averages: The statistical average \(\overline{g}\) of an arbitrary functional \(\overline{g}[u(x), v(x), w(x)]\) of the three functions is
where \( u(x, t), \ldots \) is the solution of our equations with \( u(x), \ldots \) as initial values for \( t = 0 \). If the function \( F(x) \) given in our model satisfies certain requirements (which will be fulfilled "in general") the average \( \overline{\phi} \) turns out to be essentially independent of the initial values \( u(x), v(x), w(x) \) (property of ergodicity). Of course it will depend on \( \mu \). The probability that the point \( [u(x), \ldots] \) of our function space falls into a given subset of this space is defined by the average of the "characteristic functional" of this set (1 inside, 0 outside). The statistics defined in this natural manner varies with \( \mu \). For \( \mu > \mu_1 \) it is a trivial one while, for \( \mu \) decreasing, it will be more complex. The following fact must, however, be noted. If the real part \( a(x) \) of our given function \( F(x) \) is not too smooth a function our probability distribution becomes, in the limit \( \mu \to 0 \), more and more a Gaussian or normal one. The values \( u, v, w \) of the solutions at arbitrarily given fixed points \( x_1, x_2, \ldots, x_i \) may be regarded as chance variables. If these chance variables are denoted by \( u_1, u_2, \ldots, u_k \) respectively then the probability that

\[
u_1 > a_1, u_2 > a_2, \ldots, u_k > a_k
\]

differs, as \( \mu \to 0 \), less and less (uniformly with respect to the \( a_i \)'s) from

\[
K \int_{a_1}^{a_2} \cdots \int_{a_k}^{a_k} \exp \left\{ -\frac{1}{2} \sum A_{ii} (\xi_i - m_i)(\xi_i - m_i) \right\} d\xi_1 \cdots d\xi_k,
\]

where \( K \) is a constant chosen in such a way that the expression equals 1 if each \( a_i = -\infty \). We have \( m_i = \overline{u_i} \) and the matrix \( (A_{ij}) \) is the inverse of the correlation matrix \( (\overline{u_i - m_i})(\overline{u_j - m_j}). \) For very small values of \( \mu \) the distribution is therefore nearly determined by its moments of first and second degree. In our case these moments, i.e. the mean values \( \overline{u(x)}, \overline{v(x)}, \overline{w(x)} \) and the correlation functions \( \overline{u(x)}\overline{u(x')} \) etc., are easily evaluated. Their asymptotic forms for \( \mu \to 0 \) will be investigated. Appreciable statistical interdependence is found only in points \( x, x' \) sufficiently close to each other. Approximately the correlations depend only on the mutual differences \( |x' - x| \). This is analogous to the tendency of turbulence toward spatial isotropy and homogeneity in certain hydrodynamic cases.

The asymptotic evaluation of these moments is the cardinal problem in the theory of turbulent flow in hydrodynamics. Its great mathematical difficulties apparently arise from the fact that the spatial Fourier components of the motion are interrelated with each other, in contrast to our simple model where there is no interaction between the different frequencies of the spatial Fourier pattern. The mathematical nature and the formulation of this problem will be the subject of a later paper on the foundations of statistical hydrodynamics. Still, the continued study of models of our particular kind seems not without interest to the author. There might perhaps be a starting point in devising and discussing simple models with slight interaction.
Reduction to a Four-Dimensional Problem

We use Fourier series
\[ u = \sum u_n e^{inx}, \quad z = \sum z_n e^{inx}, \quad F = \sum F_n e^{inx} \]
and corresponding notation for the Fourier coefficients of \( v, w, a, b \). The summation extends from minus to plus infinity. Making use of the identity
\[ f \circ g^* = \sum f_n g^* e^{inx} \]
and inserting into (1.0) the differential equations for the complex Fourier coefficients we obtain
\[ \dot{u}_0 = -z_0 \sigma_{-0}^* - u_0 \]
(4.0)
\[ \dot{z}_0 = z_0 u_{-0}^* + F_{-0}^* z_0 \]
and for \( n \geq 0 \)
\[ \begin{align*}
\dot{u}_n &= -z_n \sigma_n^* - n^2 \mu u_n, \\
\dot{z}_n &= z_n u_{-n}^* + F_{-n}^* z_{-n} - n^2 \mu z_n,
\end{align*} \]
(5.0)
which shows that the Fourier components belonging to different frequencies behave completely independently of each other. The equations (5.0) are all of the same type
\[ \begin{align*}
\dot{u} &= -2z - \nu u, \\
\dot{z} &= zu^* + F z - \nu z,
\end{align*} \]
(6.0) \( (F = a + ib) \)
where \( u(t), z(t) \) are the complex-valued unknowns and where \( \nu > 0 \) is a parameter. \( F = a + ib \) is a given complex constant.

(1.2) means that
\[ F_{-n} = F_n \]
(7.0)
for all \( n \). Let us simultaneously consider the system (5.0) with some \( n \) and the same system with the index \(-n\). If the initial values at \( t = 0 \) of two respective solutions coincide, i.e. if \( u_{-n} = u_n, z_{-n} = z_n \) at \( t = 0 \), they must, according to the uniqueness theorem for ordinary differential equations and according to (7.0), coincide for all \( t \). If, furthermore, such an even solution satisfies \( u_{-n} = u_n^* \), i.e. \( u_n = u_n^* \), at \( t = 0 \) then it obviously follows from the first equation (5.0) that this relation must hold for all \( t \). According to the uniqueness of the Fourier expansion, this proves a remark made in the introduction: If a solution of (1.0) at \( t = 0 \) is even in \( x \) it must be even for all \( t \); if for such an even solution \( u \) is real at \( t = 0 \) then \( u \) is real for all \( t \).
The Solutions of (4.0) and (6.0)

Relations (4.0) and (6.0) have trivial stationary solutions,

\[ u_0 = z_0 = 0 \]

and

\[ u = z = 0 \quad \text{for all } v \]

respectively. They correspond to the trivial solution \( u = z = 0 \) of (1.0). If \( a > 0 \) in \( F = a + ib \) the system (6.0) has, in the \( v \)-interval \( 0 < v < a \), a periodic solution besides,

\[ u = -a + v, \quad z = (v(a - v))^{1/2} \exp \{-ib(t + a)\} \]

for \( 0 < v < a \)

where \( \alpha \) is an arbitrary real constant. It obviously branches off the stationary solution (6.1) at \( v = a \). If, however, \( a \leq 0 \) there is no such a periodic solution for \( v > 0 \). The behavior, as \( t \to +\infty \), of all other solutions of the fundamental systems (4.0) and (6.0) is described by the following

**Lemma.** If, in (4.0),

\[ a_0 < 0 \]

every solution \( u_0, z_0 \) of (4.0) converges, as \( t \to \infty \), toward the solution \( u_0 = z_0 = 0 \).

If, in (6.0),

\[ a \leq 0 \]

every solution of (6.0), no matter what the value of \( v > 0 \), converges toward \( u = z = 0 \).

Suppose that

\[ a > 0 \]

in (6.0). If \( v > a \) every solution of (6.0) converges toward \( u = z = 0 \). If, however, \( 0 < v \leq a \), every solution of (6.0), with the exception of those where \( z = 0 \) at \( t = 0 \), converges toward the periodic solution (6.2), i.e. for every such solution there can be found an \( \alpha \) in (6.2) such that the difference of the two solutions tends to zero. The exceptional solutions tend to \( u = z = 0 \).

In this section we confine ourselves to the proof of the lemma. The first two assertions and the first part of the third one are obvious consequences of the energy equations

\[ \frac{1}{2} \frac{d}{dt} (u_0u^*_0 + z_0z^*_0) = -(u_0u^*_0 - a_0x^*_0z^*_0), \quad a_0 < 0, \]

and

\[ \frac{1}{2} \frac{d}{dt} (uu^* + zz^*) = -(vuu^* + (v - a)z^*z^*). \]
The right side of (8.0) is, for $\nu > a$ and $a > 0$, not greater than

$$-(\nu - a)(uu^* + zz^*).$$

By integration we therefore find that every solution of (6.0) satisfies ($t \geq 0$)

$$uu^* + zz^* \leq \exp \{-2(\nu - a)t\} \quad (\nu > a > 0).$$

The case $0 < \nu \leq a$ in (6.0) requires more elaborate considerations. On introducing new real variables

$$u = q + ip, \quad z = re^{i\varphi} \quad (r \geq 0)$$

(6.0) is transformed into

$$\begin{align*}
\dot{q} &= -r^2 - \nu q \\
\dot{p} &= -\nu p \\
\dot{r} &= (q + a - \nu)r \\
\dot{\varphi} &= -b - p.
\end{align*}$$

From the second and fourth equation,

$$\begin{align*}
p &= \beta e^{-\nu t}, \\
\varphi &= -(\beta/\nu) e^{-\nu t} - b(t + \alpha),
\end{align*}$$

where $\alpha, \beta$ are constants of integration. It remains to study the equations

$$\begin{align*}
\dot{q} &= -r^2 - \nu q \\
\dot{r} &= (q + a - \nu)r \\
(12.0)
\end{align*} \quad (r \geq 0).$$

To (6.2) there corresponds the stationary solution

$$q = -a + \nu, \quad r = + (\nu(a - \nu))^{1/2}$$

of (12.0). All we have to prove is that, for $0 < \nu \leq a$, every solution of (12.0), with $r > 0$ at $t = 0$, must tend to the point (13.0) in the $q$-$r$-plane. Since, for such a solution, there is always $r > 0$ we may write

$$r^2 = e^Q.$$

(12.0) transforms into

$$\begin{align*}
\dot{q} &= -e^Q - \nu q, \\
\dot{Q} &= 2(q + a - \nu).
\end{align*}$$

We have to prove that, for $0 < \nu \leq a$, every solution of (14.0) tends as $t \to +\infty$ toward the point $(q_0, Q_0)$ where

$$q_0 = -a + \nu, \quad e^{Q_0} = \nu(a - \nu).$$

\footnote*{In the case in which $a \leq 0$ this inequality stays true if $a$ is dropped in the exponential factor.}
On eliminating $q$ from (14.0) we obtain the second order equation

$$
\dot{Q} + v \dot{Q} = H'(Q)
$$

where

$$
H'(Q) = 2v(a - v) - 2e^Q = 2(e^{Q^*} - e^Q).
$$

It remains to be shown that, for $0 < v \leq a$, every solution of (15.0) has the property

$$
Q(t) \to Q_0, \quad \dot{Q}(t) \to 0 \quad \text{for} \quad t \to +\infty.
$$

Now, the function

$$
H(Q) = \int_{Q_0}^{Q} H'(x) \, dx
$$

has the properties

$$
H(Q_0) = H'(Q_0) = 0, \quad H''(Q) < 0, \quad H(Q) \to -\infty \text{ as } |Q| \to \infty
$$

for $v < a$, i.e. for $Q_0$ finite. In the case where $v = a$, i.e. where $Q_0 = -\infty$, the latter limit relation holds for $Q \to +\infty$ only. Furthermore, in any neighborhood of $Q = Q_0$ the functions $H'(Q), H''(Q)$ remain bounded. By neighborhood we mean any finite $Q$ interval around $Q_0$ if $Q_0$ is finite and any semi-infinite interval reaching to $-\infty$ if $Q_0 = -\infty$.

On multiplying (15.0) by $\dot{Q}$ and integrating we obtain

$$
\frac{1}{2} \dot{Q}^2 + v \int_{t_0}^{t} \dot{Q}^2 \, dt = H(Q) + C_1.
$$

As $H \leq 0$ we infer from this relation first that $\dot{Q}$ remains bounded and that $\int_{t_0}^{t} \dot{Q}^2 \, dt$ is finite, and second—on using (18.0)—that $Q$ stays in a neighborhood of $Q_0$. (19.0) can now be written

$$
\frac{1}{2} \dot{Q}^2 = H(Q) + \delta(t) + C_2, \quad \delta(t) \to 0 \quad \text{as} \quad t \to +\infty.
$$

On multiplying (15.0) by $\ddot{Q}$ we obtain by integration

$$
\int_{t_0}^{t} \dot{Q}^2 \, dt + \frac{v}{2} \dot{Q}^2 = \int_{t_0}^{t} \dot{Q} H'(Q) \, dt + C_3
$$

$$
= \dot{Q} H'(Q) - \int_{t_0}^{t} \dot{Q}^2 H''(Q) \, dt + C_4.
$$

Since $Q(t)$ stays in a neighborhood of $Q_0$, $H'(Q)$ and $H''(Q)$ remain bounded. It was stated already that $\dot{Q}$ stays bounded. Hence one infers that $\int_{t_0}^{t} \dot{Q}^2 \, dt$ must be finite. As $\int_{t_0}^{t} (\dot{Q}^2 + \dot{Q}^2) \, dt$ is finite there must exist a sequence $t_n \to +\infty$ such that $\dot{Q}(t_n) \to 0$ and $\dot{Q}(t_n) \to 0$. If $Q_\ast$ denotes any value of accumulation
of the $Q(t_n)$ we must, on account of (15.0), have $H'(Q_m) = 0$ which, according to (18.0), is compatible only with $Q_m = Q_0$. On inserting $t = t_n$ in (20.0) we find, taking account of $H(Q_0) = 0$, that $C_2 = 0$ and that

$$\frac{1}{2} \dot{Q}^2 - H(Q) = \delta(t), \quad \lim_{t \to +\infty} \delta(t) = 0.$$  

As $H(Q_0) = 0$ and $H < 0$ elsewhere this relation implies what we had to prove, $Q \to 0$, $H(Q) \to 0$, i.e. $Q \to Q_0$ as $t \to +\infty$.

**Behavior of the Solutions of (1.0) as $t \to +\infty$**

The following conditions will now be imposed upon the given function

$$F(x) = a(x) + ib(x),$$

(21.0)

$$a(x) = a_o + 2 \sum_n a_n \cos nx,$$

$$b(x) = 2 \sum_n b_n \cos nx.$$  

We demand that

(22.0)  

$$a_o < 0, \quad a_n > 0 \text{ for infinitely many } n$$

and that any of the $b_n$ in finite number be linearly independent with respect to integer coefficients (for instance $b_n = \kappa^n$ where $\kappa$ is a transcendental number with $0 < \kappa < 1$).

Our lemma, now, furnishes complete information about the behavior of the solutions $u, z$ of (1.0) ($\mu$ fixed) as $t \to \infty$. The Fourier coefficients of order $n$ are solutions of (6.0) where $\nu = n^2\mu$ and $F = F_n = a_n + ib_n$. If $n$ satisfies $n^2\mu > a_n$ the corresponding coefficient tends to zero. For $n^2\mu = a_n$ the same is true since in this case the periodic solution is $0$. For $n^2\mu < a_n$ (which can happen only if $a_n > 0$), however, the $n$-th coefficients $u_n(t), z_n(t)$ will, unless $z_n(0) = 0$, tend toward

$$-a_n + n^2\mu, \quad (n^2\mu(a_n - n^2\mu))^{1/2} \exp \{-ib_n(t + \alpha_n)\}$$

where $\alpha_n$ is a suitable constant (convergence in the sense expressed in the lemma). For $\mu > 0$ fixed, the latter case can only occur for finitely many values of the index $n$ (the $a_n$ are the Fourier coefficients of an absolutely integrable function and must, therefore, tend to zero). The coefficients which fall under the first case, $n^2\mu > a_n$, satisfy according to (9) and footnote 3 below the inequality

$$u_n u_n^* + z_n z_n^* \leq (u_n u_n^* + z_n z_n^*)_{t=0} \exp \{-2(n^2\mu - a_n)t\}$$

$^1a_n$ is to be omitted if $a_n \leq 0.$
for all $t > 0$. This makes it obvious that for $t \to \infty$ not only the corresponding terms of the Fourier expansion but also their sum (which is a function of $x$ and $t$) tends to zero uniformly with respect to $x$. We have hereby proved the following

**Theorem.**

Every solution $u(x, t)$, $z(x, t)$ of (1.0) ($\mu > 0$), except the solutions described right afterward, tends for $t \to \infty$ to the special solution

$$u = \sum_{n' \mu < a_n} (-a_n + n^2 \mu)e^{inx},$$

(23.0)

$$z = \sum_{n' \mu < a_n} (n^2 \mu(a_n - n^2 \mu))^{1/2}e^{inx} \exp \{-ib_n(t + \alpha_n)\}$$

with suitable real values of the $\alpha_n$, i.e. the difference between the two solutions tends to zero uniformly with respect to $x$. The exceptional solutions are precisely those for which some Fourier coefficient of $z$ with an index satisfying $n^2 \mu < a_n$ vanishes.

If only those solutions of (1.0) are considered (and we will do so in the sequel) for which $u$, $z$ are even functions of $x$ then the limit solution (23.0) is

$$u = 2 \sum_{n' \mu < a_n} (-a_n + n^2 \mu) \cos nx,$$

(23.1)

$$z = 2 \sum_{n' \mu < a_n} (n^2 \mu(a_n - n^2 \mu))^{1/2} \exp \{-ib_n(t + \alpha_n)\} \cos nx$$

where the indices are confined to positive integers.

From now on we restrict ourselves to the solutions of (1.0) with $u$, $z$ even.

Let us describe the situation brought to light by the theorem in more detail. For $\mu \geq \mu_1 = \max (a_n/n^2)$ every solution of (1.0) tends to $u = z = 0$ (the "laminar solution"). This solution exists for any $\mu$ but it is unstable for $\mu < \mu_1$. The number $N = N(\mu)$ of terms in each sum (23.1) will, according to the hypothesis (22.0), increase beyond limit as $\mu \to 0$. Let us visualize the general case in which the positive among the $a_n/n^2$ are different from each other and let us arrange these numbers in a decreasing sequence

$$\mu_1 > \mu_2 > \mu_3 > \cdots \to 0.$$  

Every time $\mu$ decreases through such a critical value $N(\mu)$ increases by one.

The limit solutions (23.1) constitute an $N(\mu)$-dimensional torus-like manifold $\mathcal{M} = \mathcal{M}(\mu)$ in our functional phase space.

---

*These solutions also tend to limit solutions which are obtained from (23.0) simply by dropping the term with that index $n$.

*According to (7), $a_n = a$, $b_n = b$. The index $n = 0$ does not occur in the sums (23.0).

*A drastic difference between our model and what is conjectured in the hydrodynamic cases is that the time periods $2\pi/b_n$ of the partial waves become longer ($b_n \to 0$) instead of shorter.
\[ u = 2 \sum_{n} (-\alpha_n + n^2 \mu) \cos nx, \]

\[ z = 2 \sum_{n} (n^2 \mu (a_n - n^2 \mu))^{1/2} e^{i n x} \cos nx \]

where the \( N \) angular variables \( \varphi_n \) are the parameters. The limit solutions are

\[ \varphi_n = -b_n(t + \alpha_n), \]

\( \alpha_n \) being arbitrary real constants. Since the \( b_n \) were supposed to be linearly independent numbers each of those solutions will be everywhere dense on the manifold \( \mathcal{M} \).

The manifold \( \mathcal{M}(\mu) \) varies continuously with \( \mu \) though not analytically in the phase space of our problem (the function space of the \( u(x), z(x) \)). For \( \mu > \mu \), it is simply the point \( u = z = 0 \). As \( \mu \) decreases and passes \( \mu_1 \), \( \mathcal{M} \) branches off this point as a small and gradually enlarging closed curve. On passing \( \mu_2 \), \( \mathcal{M} \) branches again off this curve and forms a thin and gradually swelling tire. The curve continues to exist but it has yielded its stability to the tire. As \( \mu \) passes through \( \mu_a \) another branching occurs and so forth. The number \( \mathcal{N}(\mu) \) of dimensions of \( \mathcal{M}(\mu) \) (the "number of degrees of freedom of turbulence") increases beyond limit.

**Limit of the Solutions of (1.0) for \( \mu \to 0 \)**

If the initial values \( u(x), z(x) \) from which a solution of our model, with \( \mu \) given, starts are chosen at random then the solution actually observed in the long run will be precisely a solution (23.1). Passing to the limit \( \mu \to 0 \) in the observed solution therefore means letting \( \mu \to 0 \) in (23.1).

We suppose (only in this section) that \( F(s) \) satisfies the following condition (which could, however, be replaced by a less restrictive one)

\[ \sum |a_n| < \infty. \]

Under this condition we can easily prove:

As \( \mu \to 0 \) the functions (23.1) converge towards the time independent functions

\[ u = -2 \sum_{a_n > 0} a_n \cos nx, \quad z = 0 \]

uniformly with respect to \( x \) and \( t \).

The turbulent fluctuations which, in our model, occur only in the \( z \)-component disappear in the limit \( \mu \to 0 \). Incidentally (26.0) is a special stationary solution of the equations (1.0) with \( \mu = 0 \). These equations have infinitely many stationary solutions: every pair \( u = u(x), z = 0 \) where \( u \) has a vanishing mean value is such a solution for \( \mu = 0 \).

The proof of the limit relation is simply carried out by splitting each sum in (23.1) into two sums.
\[ \sum = \sum_{n \leq k} + \sum_{n > k} \]

with \( k \) fixed. In the sums of the first type \( \mu \to 0 \) can be effected without difficulty. The absolute value of the sums of second type is less than \( \sum |a_n| \) which can, by choosing \( k \) sufficiently large, be made arbitrarily small\(^1\).

The Statistics in Phase Space

We consider the non-trivial case \( \mu < \mu_1 \) in our problem. Let \( \mathcal{F}[u(x), z(x)] \) be an arbitrary functional. \( \mathcal{F} \) is supposed to vary continuously if the argument functions (and their first derivatives) change uniformly continuously.\(^*\) If \( [u(x), z(x)] \) does not belong to the exceptional set as stated in our main theorem the difference between \( f(t) = \mathcal{F}[u(x, t), z(x, t)] \), where \( u(x, 0) = u(x), z(x, 0) = z(x) \), and the same function, where a suitable solution (23.1) is substituted, tends to zero as \( t \to \infty \). Therefore,

\[
\overline{\mathcal{F}} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathcal{F}[u(x, t), z(x, t)] \, dt
\]

(27.0)

\[
= \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathcal{F}_{a, s = (23.1)} \, dt
\]

if the latter limit exists. The statistics is reduced to a statistics on the restricted finite-dimensional manifold (24.0) of solutions. On \( \mathcal{M} \) the functional \( \mathcal{F} \) is simply a continuous function \( f(\varphi_1, \cdots, \varphi_N) \) of the parameters \( \varphi \), with period \( 2\pi \) in each of them. According to (25.0) where the \( b_n \) are linearly independent and according to the equidistribution theorem of H. Weyl the right hand limit in (27.0) exists and is independent of the initial phases. The phase flow on \( \mathcal{M} \) is ergodic. It follows that the time average of any continuous functional \( \mathcal{F}[u(x), z(x)] \) has, for the majority of initial phases, the constant value

\[
\overline{\mathcal{F}} = \frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathcal{F}_{a, s = (24)} \prod_{n \in \mathbb{Z}} d\varphi_n .
\]

(28.0)

For instance, the averages of polynomial functionals of degree one and two are determined by \((z = v + iw)\)

\[
\overline{v(z)} = \overline{u(z)} = 0
\]

(29.0)

and\(^*\)

\[
\overline{v(z)w(z')} = 0, \quad \overline{v(z)w(z')} = \overline{w(z)w(z')} = h_a(x + x') + h_a(x - x')
\]

(30.0)

\(^1\)In the second sum of (23.1) use the inequality \( v(a - v) \leq \frac{1}{a} \).

\(^*\)Instances of such functionals are \( z(z_1), z(z_1)z^*(z_2) \), where \( z_1, z_2 \) are fixed points, and

\[
\int_0^\tau z(z)z^*(z) \, dz, \quad \int_0^\tau \frac{dz(z)}{dz} \frac{dz^*(z)}{dz} \, dz .
\]

\(^*\)\( u \) can be left out of consideration since, on \( \mathcal{M} \), \( u \) is independent of the time, \( u = \overline{u} \).
where

\[ h_\mu(y) = \sum_{n^2 \mu - n^2 \mu} n^2 \cos ny. \]

In (29.0) and (30.0) \( x, x' \) are arbitrary fixed points in \((0, \pi)\). The functionals considered in (30.0) (products of values of \( u, v, w \) at fixed points) are the simplest of degree two. Every other quadratic functional can be written as a linear functional of the functions (30.0) of \( x, x' \).10

**Asymptotic Form of the Correlations for \( \mu \to 0 \)**

In order to avoid too lengthy considerations we will, from now on, restrict the given function \( a(x) \) to the class of functions \((a, < 0)\)

\[ a(x) = a_0 + \sum n^{-s} \cos nx \quad (s > 0). \]

All these functions are absolutely integrable. We shall first show that if

\[ \sum n^2 a_n \text{ diverges, i.e. } s \leq 3, \]

the correlations (30.0) have in the interval \((0, \pi)\) a tendency toward homogeneity as \( \mu \to 0 \). The main fact is that

\[ \lim_{\mu \to 0} \frac{h_\mu(y)}{h_\mu(0)} = 0, \quad 0 < y < 2\pi \]

holds uniformly in any closed subinterval of \((0, 2\pi)\). This would obviously no longer be true if \((s > 3) \sum n^2 a_n \text{ converges, i.e. if the function } a(x) \text{ is too smooth. The limit behavior, for } \mu \to 0, \text{ of the correlation quantities is now evident from (30.0). The dispersions about the mean values } \bar{v} = \bar{w} = 0 \]

\[ \bar{v}(x) = \bar{w}(x) = h_\mu(0) + h_\mu(2x) \]

are proportionally nearly constant in the fundamental interval \(0 < x < \pi\).

At its endpoints there is a sharp rise to the double value. The coefficient of correlation at two points \( x, x' \in (0, \pi) \)

\[ \frac{v(x)v(x')}{(\bar{v}(x))^{1/2}(\bar{v}(x'))^{1/2}} = \frac{w(x)w(x')}{(\bar{w}(x))^{1/2}(\bar{w}(x'))^{1/2}} \sim \frac{h_\mu(x' - x)}{h_\mu(0)} \]

has appreciable values only for sufficiently small distances \(| x' - x | \). The statistical distributions at two not too close points are nearly independent of each other.

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10In the hydrodynamic case of the flow through a channel with a given net flow through a cross section, for instance, the force exerted by the fluid on the wall is a second degree functional of the instantaneous velocity field. It can be computed, since the correlations (of second degree) between the velocities at different points are known functions of these points.
Relation (34.0) will be proved further below. An asymptotic expression for

\[ h_s(y) = \mu \sum_{n=1}^{N} n^2(\mu - n^2) \cos ny \]

where \( N \) is the largest integer less than \( \mu^{-1/(2+s)} \) is easily found by setting \( y = \eta/N \) where \( \eta \) is kept fixed and by inserting \( \mu = (N + \delta)^{-2-s} \) where \( 0 < \delta < 1 \). The sum (35.0) can then be written as a Riemann approximating sum for a definite integral. The asymptotic expression, for \( \mu \) small, is

\[ h_s(y) \sim N^{2s-1} \int_0^1 \alpha^2(\alpha^{-s} - \alpha^2) \cos N\alpha \, d\alpha, \quad N \sim \mu^{-1/(2+s)}. \]

This shows that appreciable correlation occurs only at distances of the order of \( N^{-1} \sim \mu^{1/(2+s)} \). The case \( s = 3 \) where the asymptotic behavior is different has been excluded here. (36.0) implies that

\[ h_s(0) \sim \frac{2 + s}{5(3 - s)} \mu^{(2s-1)/(s+2)}. \]

This shows, in extension of the result of a previous section, that the turbulent fluctuations die down for \( \mu \to 0 \) if \( s > 1/2 \). For \( s < 1/2 \), however, they are seen to increase beyond limit.

Relation (34.0) is proved by application of Abel's partial summation to the sum in the denominator (35.0) of (34.0). The transformed sum is a sum of products where one factor is a partial sum of \( \sum \cos ny \) and where the other factor is the difference of two successive coefficients in the Fourier sum (35.0). Those partial sums are known to be uniformly bounded in any closed subinterval of \((0, 2\pi)\). On the other hand, the sum of the absolute values of those differences is easily found to be bounded if \( s \geq 2 \) and to be of the order of \( N^{2-s} \) if \( s < 2 \). The denominator sum in (34.0), however, is of the order of \( N^{3-s} \) for \( s < 3 \) and of the order of \( \log N \) for \( s = 3 \). This proves (34.0).

There is no reason to investigate this matter any further because we have, now, arrived at a point at which the analogy to the hydrodynamic case breaks down. The limiting form of the correlation curves in our model case is quite different from the one in hydrodynamics. Recent investigations by Kolmogoroff, von Weissäcker, and Heisenberg indicate that the principal correlation coefficient in hydrodynamics has approximately the universal form

\[ 1 - \text{const.} \cdot |x' - x|^{2/3} \]

if the distance is small compared to the dimensions of the boundaries and large as compared to a length depending on the viscosity \( \mu \) and tending to zero with \( \mu \). Their reasoning essentially uses the fact that there is interaction between the different frequencies of the spatial Fourier pattern (which is entirely absent in our case). The arguments are based on the Prandtl mixing length or similar highly intuitive concepts. These notions in turn rest on a semi-
empirical picture of turbulent fluid motion. The ultimate goal, however, must be a rational theory of statistical hydrodynamics where those important results and other properties of turbulent flow can be mathematically deduced from the fundamental equations of hydromechanics.

**Normality of the Distribution**

In the hydrodynamic case we are to expect that the values of the velocities in an arbitrarily given finite set of fixed points at rest become more and more normally distributed as \( \mu \to 0 \). In our model case this can be actually proven.

The values \( w(x_1), \ldots, w(x_j) \) at \( j \) fixed points \( x_1, \ldots, x_j \) in the open interval \((0, \pi)\) may be regarded as a set of chance variables the distribution law of which has been determined before. We use the parametric representation

\[
(24.0) \quad \begin{align*}
    u(z) &= c_P(z), \\
    w(x) &= c_W(x)
\end{align*}
\]

where

\[
(n) \quad \begin{align*}
    \left\{ \begin{array}{c}
        v(x) = \sum_{n^2 \mu < a^2} v(n)(x), \\
        w(x) = \sum_{n^2 \mu < a^2} w(n)(x)
    \end{array} \right.
\]

and where the \( \varphi_n \) are the fundamental chance variables with the probability differential element

\[
\prod \left( d\varphi_n / 2\pi \right).
\]

If we regard the set of values

\[
(39.0) \quad (v(x_1), w(x_1), \ldots, w(x_j))
\]

as a chance vector we see that this vector is the sum of certain chance vectors

\[
(40.0) \quad (v(n)(x_1), w(n)(x_1), \ldots, w(n)(x_j))
\]

where the summation index runs through all values that satisfy \( n^2 \mu < a^2 \). Now, these different chance vectors are obviously statistically independent of each other which means that any set of components where no two of them belong to the same vector constitutes a set of independent chance variables. The components of one and the same vector (40.0) may, however, be statistically correlated among each other. As the number of terms \( \chi(0) \) in our sum tends \( \to \infty \) for \( \mu \to 0 \) we should expect the probability distribution of the variables (39.0) (with points \( x_1, \ldots \) arbitrarily fixed in advance) to deviate less and less from a multidimensional normal distribution as formulated above in the introduction. The normal distribution will, in this case, be the one with the same first and second degree moments (mean values and correlations) as the actual distribution. The central limit theorem of probability theory affirms
the truth of this limit relation provided that the following two conditions are satisfied. The first one roughly says that none of the vectors \((40.0)\) plays too dominant a statistical role in their sums \((39.0)\). The second condition means that the correlation hyperellipsoid of the variables \((39.0)\) does not degenerate as \(\mu \to 0\). The first condition (this is the well known Lindeberg condition) splits up into a set of conditions referring to the single components of \((39.0)\) i.e. to the chance variable \(v(x)\) or to \(w(x)\) in \((37.0), (38.0)\) with a fixed point \(x\) in \((0, \pi)\). As all the terms in a sum \((38.0)\), apart from the values of their first and second degree moments, have the same distribution the Lindeberg condition on the sum \(v(x)\) or \(w(x)\) simply reduces to the condition that, for \(\mu \to 0\),

\[
\frac{(v^n(x))^2}{(v(x))^2} = \frac{(w^n(x))^2}{(w(x))^2} = \frac{n^2(a_n - n^2\mu) \cos^2 n x}{\sum_{a_i > a_n} i^2(a_i - i^2\mu) \cos^2 i x} \quad (a_n > n^2\mu)
\]

converge to zero uniformly with respect to \(n\).

As in the preceding section we now restrict ourselves to the case where the given function \(a(x)\) is of the special form \((32.0)\). We shall verify that the Lindeberg condition and the dimensionality condition are fulfilled if

\[ \sum n^s a_n \text{ diverges, } s \leq 3, \]

i.e. under the same hypothesis as in the preceding section. The dispersion ratios in question are not greater than

\[ \frac{n^2(n^{-2} - n^2\mu)}{\sum_i i^2(i^{-2} - i^2\mu) \cos^2 i x} \quad (n \leq N) \]

respectively where \(N(\mu)\) is the largest integer \(<\mu^{-1/(2s+1)}\). The denominator is \(\frac{1}{4}h_s(0) + \frac{1}{4}h_s(2\pi)\). According to \((34.0)\) for the purpose of investigating the order of magnitude the cos-factors in the denominator may be dropped. Now, in the preceding section, we have studied the ratio where the denominator was the same but where the denominator was the sum of the absolute values of the differences of successive denominators in \((42.0)\). That ratio was found to tend to zero for \(\mu \to 0\) if \((41.0)\) is fulfilled. For \(n = 1 \) \((42.0)\) obviously tends \(\to 0\) under condition \((41.0)\). Hence we infer that \((42.0)\) a fortiori tends to zero uniformly with respect to \(n\).

As to the dimensionality condition we need only observe that the correlation matrix of the chance variables \((39.0)\), \(x_1, \cdots, x_i\), being fixed points, becomes, for \(\mu \to 0\), more and more the unit matrix times a scalar factor, i.e. that the correlation hyperellipsoid becomes a hypersphere if the condition \((41.0)\) is satisfied. It was manifestly this fact which resulted from the preceding section.

The distribution of the values at arbitrarily preassigned fixed points \(x_1, \cdots, x_i\) in \((0, \pi)\) becomes, for \(\mu \to 0\), a spherical normal distribution.

The result can, of course, be generalized by dropping the condition that the points \(x_i\), be kept fixed while \(\mu \to 0\). If the points \(x_1, \cdots, x_i\) \((j \text{ fixed})\) are
allowed to vary while $\mu \to 0$ the only (sufficient) condition to be imposed on them will be that the distances from 0 and $\pi$ and between any two of them stay greater than $\text{const.}/N(\mu)$ while $\mu \to 0$. This was found to be the order of distance with appreciable correlation. In this general case, of course, the correlations will enter in the limit distribution.

The fundamental reason why we obtained a normal distribution in the limit was the fact that in our model the components of the spatial Fourier pattern behave statistically independently. In the hydrodynamic case where we have interaction the dependence will probably be only slight in the sense that it will not extend far over the spatial spectrum. With such an approximate independence the central limit theorem of probability theory might still hold. The distribution of the simultaneous velocities at a fixed finite set of space points might be expected to become normal. It is, however, not certain if the space derivatives $\partial u/\partial x$, \cdots also behave like this. It is quite possible that their distribution does not approach a normal one.