Constant-eddy-viscosity models of vertical structure forced by periodic winds

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Abstract—We consider the vertical current structure forced by a periodic surface wind acting on a homogeneous, constant-eddy-viscosity ocean. The vertical dynamics are described by a simple extension of Ekman's (1905) steady-wind solution. At forcing frequencies other than the inertial frequency, the energy from the surface wind stress propagates down into the water column as a pair of counter-rotating waves. At the inertial frequency, which is the frequency of free waves of the system, the forced current profile is quadratic in the depth, and, for a given wind stress, the magnitude of the current is proportional to the water depth. Viewed in a reference frame that is fixed in space rather than fixed to the earth's surface, the inertial waves are not wavelike at all, but are steady in time. Interpretation of the dynamics at frequencies other than inertial is also more straightforward in a fixed reference frame.

1. INTRODUCTION

Discussions of vertical current structure tend, almost inevitably, to begin with Ekman's (1905) solution for the current profile in a well-mixed, constant-eddy-viscosity ocean of infinite extent, forced by a steady surface wind. Ekman's classical solution shows the current starting at the surface at an angle of $\frac{1}{4}\pi$ to the wind direction, spiralling in a clockwise direction (in the northern hemisphere) down through the water column, under the combined action of shear stress and Coriolis force.

A constant eddy viscosity is not generally regarded as a good model for the ocean (e.g. Csanady and Shaw, 1980). There have been many variations on Ekman's (1905) study, using different representations of the eddy viscosity as a function of depth; see for example, the discussion in Jordan and Baker (1980). In general, for other representations the spiral structure of the currents remains, but details of the spiral are sensitive to the value of the eddy viscosity.

The wind stress, in Ekman's study, was impulsively applied to start the system from rest, and surface gradients were ignored. The solution was then a superposition of transient inertial waves and the steady state (GONELLA, 1971). One extension of Ekman's theory followed from Welander's (1957) observation that surface gradients can be treated as a forcing function similarly to the wind stress. If the time histories of the wind stress and the surface gradients are known, then the vertical profile can be calculated at any location as the sum of vertical eigenfunctions, with the time-dependence given by an inverse Laplace transform (Jelesnianski, 1970). This solution procedure was developed by Jelesnianski (1970) and Forristall (1974) for a constant eddy viscosity, and

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subsequently extended to vertically variable eddy viscosity by Nihoul (1977), Jordan and Baker (1980) and Davies (1987, 1988). In this form of study, the surface gradients are generally calculated by a conventional depth-averaged numerical circulation model, with the vertical calculation used either to give the vertical current profiles, or to improve the representation of the bottom stress.

Model formulations in which the depth-averaged and depth-varying currents are time-integrated by the two different techniques are now commonly described as two-and-a-half dimensional. The model becomes fully three dimensional with the (numerical) integration of the vertical eigenfunction amplitudes at every time-step of the depth-averaged model, so that both depth-averaged and depth-varying currents can be used in estimation of the bottom stress. Heaps (1972) pioneered three-dimensional eigenfunction models, and recent developments are described by Davies and Stephens (1983) and Davies (1985). Even in the three-dimensional model it can still be advantageous to separate the depth-averaged and depth-varying calculations, both for the sake of ease of interpretation, and numerical efficiency (Davies, 1985).

The fully three-dimensional numerical model is, of course, considerably more complicated to implement than Ekman's (1905) original model, and its results are correspondingly more difficult to interpret. To begin with, in a numerical simulation, there are likely to be many time scales, while in Ekman's formulation there was only one, the inertial period.

An intermediate model is obtained by a second extension of the Ekman formulation, the introduction of a single frequency, periodic forcing. This approach is particularly relevant to the continental shelf, where important forcing influences may have time scales comparable with the inertial period. An oscillatory formulation is obviously appropriate to tidal modelling; it appears to have been first used in this context by Sverdrup (1927). Recently, Tee (1979), Prandle (1982a,b), Soulsby (1983) and Maas and van Haren (1987) have examined tidally induced vertical structure from this point of view. Studies of periodic wind forcing have been more limited, but of note is Gonella's (1972) "Ekman theory extension" introduced in the context of rotary-component analysis of currents. Webster (1986) has applied an extension of Gonella's (1972) technique to the Australian North West Shelf.

A single-frequency wind may be regarded as a dominant periodic wind cycle, such as a sea-breeze, or as a single Fourier component in the continuous wind spectrum. A linear model, forced by such a spectral component of the wind, will yield the spectral component of the current at the same frequency, so that modelling in the time and frequency domains are equivalent.

In an eddy-viscosity model, the vertical structure of the periodic motion is governed by an ordinary differential equation, similar to that in Ekman's study. Sverdrup (1927) first showed that the solution is most naturally viewed as a pair of counter-rotating current vectors. There is a different depth scale associated with each direction of rotation and, as the frequency tends to zero, each of these scales approaches the Ekman depth.

Any periodic motion of a two-dimensional vector can be expressed as the sum of two counter-rotating vectors. As inferred by Gonella (1972), under periodic wind forcing, each rotating current vector is a response to the wind vector rotating in the same direction. The dynamics of the rotating current components are examined in detail in Sections 2 and 3 of the present paper.

When the wind forcing has the inertial period, the solution breaks down. The depth-
scale and the current amplitude appear to go to infinity. Since the inertial frequency is the frequency of the free waves of the system (GONELLA, 1971), it is tempting to describe this situation as resonance. However, as MAAS and VAN HAREN (1987) point out, an inertial oscillation is more properly viewed as a non-rotating solution. The form of both the free waves and the forced inertial wave are described in Sections 4 and 5 of the present paper. We then note, in Section 6, that explanation of the dynamics at all frequencies, including the inertial, is more straightforward in a non-rotating frame of reference.

The discussion throughout the study is limited to an eddy viscosity constant in time and space. Interestingly, MAAS and VAN HAREN (1987) found that a constant eddy viscosity gave a good description of the current profiles at the dominant tidal frequencies in the North Sea. In general, however, inadequate knowledge of the eddy viscosity remains a major obstacle to the accurate prediction of vertical current structure. Nevertheless, the constant eddy viscosity solutions, because they are simple and analytic, provide a valuable introduction to the realistic dynamics, just as in the case of the original EKMAN (1905) solution.

2. FORMULATION

The governing equations are the standard linear momentum and continuity equations for a well-mixed hydrostatic ocean; that is

\[ u_t -fv = -g \zeta_x + \mu u_{zz}, \]  (1)

\[ v_t +fu = -g \zeta_y + \mu v_{zz}, \]  (2)

\[ \zeta_t + \frac{\partial}{\partial x} \int_{-H}^{0} u dz + \frac{\partial}{\partial y} \int_{-H}^{0} v dz = 0, \]  (3)

expressed in Cartesian coordinates x, y and z, in which z is vertically upwards and u and v are the horizontal velocity components. The surface displacement, from the mean position z = 0, is \( \zeta \) and the sea bed is given by z = -H. Constants in (1), (2) and (3) are the Coriolis parameter f, which will be assumed positive, the gravitational acceleration g and the eddy viscosity \( \mu \). Time is represented by t, and alphabetic subscripts represent differentiation.

It may immediately be seen that, if the surface gradients \( \zeta_x \) and \( \zeta_y \) are known, (1) and (2) form a pair of coupled parabolic equations, with no explicit x and y dependence. This observation forms the basis for the two-and-a-half dimensional models, described in Section 1, by JELESNIANSKI (1970) and later authors.

Surface and bottom boundary conditions appropriate to (1) to (3) are as follows. At the surface, the stress is given by the wind stress \( \rho \tau, \rho \tau \); that is

\[ \rho \mu u_z = \tau_x \text{ and } \rho \mu v_z = \tau_y, \text{ at } z = 0, \]  (4)

where \( \rho \) is the water density. At the bottom, the stress is set proportional to the bottom current:

\[ \mu u_z = ru_b \text{ and } \mu v_z = nv_b, \text{ at } z = -H, \]  (5)

where \( r \) is a constant, having units of velocity, and \( u_b \) and \( v_b \) are the velocity components \( u \) and \( v \) evaluated at \( z = -H \).
In both analytic and numerical studies, it is convenient to separate the dynamics into depth-averaged and depth-varying flow. If $U$ and $V$ represent the depth averages of $u$ and $v$, then

\begin{align}
U_t - fV &= -g \frac{\zeta_x}{H} + \frac{\tau^x}{\rho H} - \frac{r}{H} u_b, \\
V_t + fU &= -g \frac{\zeta_y}{H} + \frac{\tau^y}{\rho H} - \frac{r}{H} v_b,
\end{align}

and

\begin{align}
\zeta_t + (HU)_x + (HV)_y &= 0.
\end{align}

The equations for the depth-varying component of the flow, denoted by $u'$ and $v'$, are obtained by subtracting (6) and (7) from (1) and (2) to give

\begin{align}
u'_t - fv' &= \mu u''_{zz} - \frac{\tau^x}{\rho H} + \frac{r}{H} u_b, \\
v'_t + fu' &= \mu v''_{zz} - \frac{\tau^y}{\rho H} + \frac{r}{H} v_b.
\end{align}

The flow component $(U, V, \zeta)$ is described by Csanyi (1982) as the "global solution", and $(u', v')$ as the "local solution". The free-wave solutions to (6), (7) and (8) are surface inertial-gravity waves, while the free-wave solutions to (9) and (10) are strict inertial waves (e.g. GONELLA, 1971; CRAIG, 1988). In general, the surface wave period is an order-of-magnitude shorter than the inertial wave period so that, in numerical studies, the time-step required to resolve the global solution is much smaller than that for the local solution. Thus, in such studies, the solutions are often considered separately for the sake of efficiency as well as ease of interpretation (e.g. DAVIES, 1985). It should be noted, however, that the global and local solutions are not independent of one another since the bottom velocity, containing both components, appears in both sets of equations.

In the present study, we shall concentrate on the local solution, examining in particular the dynamics forced by a steady oscillating wind. For mathematical convenience, variables will be treated as complex, so that the form of the wind stress is assumed to be given by

\begin{align}
\frac{\tau^x}{\rho} &= X e^{-i\omega t} \quad \text{and} \quad \frac{\tau^y}{\rho} = Y e^{-i\omega t},
\end{align}

where $X$ and $Y$ are constant amplitudes, and the frequency $\omega$ is fixed, positive and real. As mentioned in Section 1, the wind stress in (11) may be envisaged as that due to a single dominant wind pattern, such as a sea breeze, or as a single constituent of the spectrum, from which the full time-series of the wind may be assembled.
Neglecting transients, the solution to (9) and (10) may now be written as

\[
\begin{align*}
    u' &= \frac{1}{(f^2 - \omega^2)H} \left\{ i\omega \left( \frac{\tau^x}{\rho} - ru_b \right) + f \left( \frac{\tau^y}{\rho} - rv_b \right) \right\} \\
    &\quad + \{ a_1 \exp(\alpha_1 z) + a_2 \exp[-\alpha_1 (z + H)] + c_1 \exp(\alpha_2 z) + c_2 \exp[-\alpha_2 (z + H)] \} e^{-i\omega t} \\
\end{align*}
\]

(12)

and

\[
\begin{align*}
    v' &= \frac{1}{(f^2 - \omega^2)H} \left\{ f \left( \frac{\tau^x}{\rho} - ru_b \right) + i\omega \left( \frac{\tau^y}{\rho} - rv_b \right) \right\} \\
    &\quad + \{ i a_1 \exp(\alpha_1 z) + i a_2 \exp[-\alpha_1 (z + H)] - i c_1 \exp(\alpha_2 z) - i c_2 \exp[-\alpha_2 (z + H)] \} e^{-i\omega t},
\end{align*}
\]

(13)

where

\[
\begin{align*}
    \alpha_1 &= (1 - i) \left( \frac{\omega + f}{2\mu} \right)^\dagger, \\
    \alpha_2 &= (1 - i) \left( \frac{\omega - f}{2\mu} \right)^\dagger \quad \text{if } \omega > f, \\
    &= (1 + i) \left( \frac{f - \omega}{2\mu} \right)^\dagger \quad \text{if } \omega < f,
\end{align*}
\]

(14)

and \( a_1, a_2, c_1 \) and \( c_2 \) are constant amplitudes to be determined from the boundary conditions.

We observe initially that each of \( u' \) and \( v' \) in (12) and (13) consists of a depth-constant part, which ensures that the depth integral is zero, and four vertically propagating waves. The waves with amplitudes \( a_1 \) and \( c_1 \) decay away from the surface, while those with amplitude \( a_2 \) and \( c_2 \) decay away from the bottom. When \( \omega = 0 \), there is only one decay scale, the Ekman depth, given by \((2\mu/f)^\dagger\). However, for nonzero \( \omega \) there are two distinct depth scales. When \( \omega = f \), the solution given by (12) to (14) is invalid.

3. THE VERTICALLY PROPAGATING WAVES

The two waves with amplitudes \( a_1 \) and \( c_1 \) may be written separately as

\[
\begin{align*}
    u^a &= a_1 \exp(\alpha_1 z - i\omega t), \\
    v^a &= ia_1 \exp(\alpha_1 z - i\omega t),
\end{align*}
\]

(15)
and

\[
\begin{align*}
\vec{u} & = c_1 \exp(a_2 z - i\omega t), \\
\vec{v} & = -ic_1 \exp(a_2 z - i\omega t).
\end{align*}
\] (16)

At a fixed value of \(z\), the velocity vector for the wave given by (15) has a constant magnitude and rotates anticlockwise in time with frequency \(\omega\). The velocity vector for the wave given by (16) also has a constant magnitude at a particular \(z\), but rotates clockwise in time. The two will thus be described as the anticlockwise and the clockwise waves, respectively; it should be borne in mind that this directionality refers to time and not space.

The behaviour in space of the two waves can be seen in the hodographs in Fig. 1. Moving down through the water column at a fixed time, the velocity vector for the anticlockwise wave (15) rotates clockwise (Fig. 1a). For the clockwise wave (16) the velocity vector rotates anticlockwise with depth if \(\omega > f\) (Fig. 1b), but clockwise if \(\omega < f\) (Fig. 1c).

The hodograph in Fig. 1a simply rotates anticlockwise in time, while those in Fig. 1b and c rotate clockwise.

The manner in which the two waves are generated by the wind may be determined by referring to the surface boundary condition (4). We ignore for the present the effects of the bottom boundary by setting \(a_2 = c_2 = 0\) in (12) and (13), so that condition (4) is sufficient to determine \(a_1\) and \(c_1\). Consider first an anticlockwise rotating wind stress given by

\[
\vec{\tau} = \tau^a, \quad \vec{\tau'} = \tau^a.
\]

Then, solving for \(a_1\) and \(c_1\) in (12) and (13), using (4), gives

\[
a_1 = \frac{\tau^a}{\rho \mu a_1}, \quad c_1 = 0.
\] (17)

Similarly, if the wind stress rotates clockwise, that is

\[
\vec{\tau} = \tau^c, \quad \vec{\tau'} = -i\tau^c,
\]

then

\[
a_1 = 0, \quad c_1 = \frac{\tau^c}{\rho \mu a_2}.
\] (18)

Thus, an anticlockwise rotating wind generates the anticlockwise wave, while a clockwise wind generates the clockwise wave.

The amplitude of the wave generated by a clockwise wind stress will be larger, by a factor of \([((\omega + f)/|\omega - f|)]\), than that generated by an anticlockwise wind stress. For the anticlockwise wave, and the clockwise wave when \(\omega < f\), the surface current is at an angle of \(\pi/4\) clockwise to the wind direction. For the clockwise wave when \(\omega > f\), the surface current is at \(\pi/4\) anticlockwise to the wind direction.

The properties of the two waves are summarized in Table 1. For \(\omega\) close to \(f\), the clockwise wave has a much longer wavelength than the anticlockwise wave; the wavelength is the depth over which a full rotation of the current vector occurs. The
Fig. 1. Instantaneous hodographs of the current vector, calculated with $f = 10^{-4}$ $s^{-1}$, $\mu = 0.01$ $m^2 s^{-1}$: (a) the anticlockwise wave, $\omega = 1.2 \times 10^{-4}$ $s^{-1}$; (b) the clockwise wave, $\omega = 0.8 \times 10^{-4}$ $s^{-1}$, i.e. $\omega < f$; (c) the clockwise wave, $\omega = 1.2 \times 10^{-4}$ $s^{-1}$, i.e. $\omega > f$. Tick marks are at 20 m intervals. Current amplitudes are given by (17) and (18), for a wind stress of 0.1 N $m^{-1}$ instantaneously directed along the x-axis.
Table 1. A summary of the properties of the two vertically propagating waves

<table>
<thead>
<tr>
<th>Property</th>
<th>Anticlockwise wave</th>
<th>Clockwise wave</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotation in time</td>
<td>$a - c/w$</td>
<td>$c/w$</td>
</tr>
<tr>
<td>Rotation with depth</td>
<td>$c/w$</td>
<td>$c/w$ ((\omega &lt; f)) $a - c/w$ ((\omega &gt; f))</td>
</tr>
<tr>
<td>Angle from wind vector to surface current vector</td>
<td>$\frac{\pi}{4} a - c/w$ ((\omega &lt; f))</td>
<td>$\frac{\pi}{4} a - c/w$ ((\omega &gt; f))</td>
</tr>
<tr>
<td>Wavelength</td>
<td>$2\pi \left(\frac{2\mu}{\omega + f}\right)^\dagger$</td>
<td>$2\pi \left(\frac{2\mu}{\omega - f}\right)^\dagger$</td>
</tr>
<tr>
<td>Vertical decay scale</td>
<td>$\left(\frac{2\mu}{\omega + f}\right)^\dagger$</td>
<td>$\left(\frac{2\mu}{\omega - f}\right)^\dagger$</td>
</tr>
<tr>
<td>Group velocity</td>
<td>$-\omega \left(\frac{2\mu}{\omega + f}\right)^\dagger$</td>
<td>$-\omega \left(\frac{2\mu}{\omega - f}\right)^\dagger$</td>
</tr>
<tr>
<td>Phase speed</td>
<td>$-\omega \left(\frac{2\mu}{\omega + f}\right)^\dagger$</td>
<td>$\left(\frac{\omega}{f - \omega}\right) \left(2\mu \frac{\omega - f}{\omega}\right)^\dagger$</td>
</tr>
</tbody>
</table>

The anticlockwise wave decays more rapidly with depth than the clockwise wave. The group velocity for both waves is positive downwards. The energy is injected at the surface by the wind stress and propagates down through the water column. The phase velocity is also positive downwards for the anticlockwise wave, and the clockwise wave when \(\omega > f\), but upwards for the clockwise wave when \(\omega < f\).

As noted by Gonella (1972), the wave properties, as listed in Table 1, are continuous at \(\omega = 0\). That is, the properties of the anticlockwise wave may be obtained from those of the clockwise wave by replacing \(\omega\) with \(-\omega\). There is discontinuity at \(\omega = f\); the reason for this will be examined in the following sections.

To this stage, the waves having amplitude \(a_2\) and \(c_2\) in (12) and (13) have been ignored. These are the waves that result from reflection, from the bottom, of the two waves with downward energy propagation. Properties of the reflected waves may be derived in the same manner as for the incident waves. The reflected waves, as is to be expected, carry energy upward through the water column. The amplitude of the reflected wave is determined by the bottom boundary conditions (5).

From (5), it follows that an anticlockwise wave with amplitude \(a_1\) will be reflected as an anticlockwise wave with amplitude \(a_2\) [in the notation of (12) and (13)] given by

\[
a_2 = \frac{\mu a_1 - r}{\mu a_1 + r} a_1 e^{-a_1 \mu z}.
\]

If the bottom friction coefficient, \(r\), is zero, the wave is reflected with no change of amplitude or phase. If \(r\) is non-zero, there is an amplitude reduction and a phase change. For the clockwise wave, a relationship similar to (19) holds. Each of the incident waves reflects as a wave that rotates in the same direction in time, but in the opposite direction with \(z\).

Of course, in a full analysis, one does not first derive the amplitudes \(a_1\) and \(c_1\) followed by the reflected amplitudes \(a_2\) and \(c_2\). All four would need to be derived simultaneously, using boundary conditions (4) and (5) and allowing, in shallow water at least, for multiple reflections from both surfaces.
A full analysis would also require investigation of the "global" as well as the "local" dynamics. The global equations, (6) to (8), contain lateral gradients, and their solution requires consideration of the lateral structure of the bathymetry and the wind forcing. It is worth noting, however, that over the continental shelf, the current profiles will often be dominated by the local rather than the global dynamics. This result follows from (6) and (7) because, for the depth-averaged flow, the wind stress acts as body force that is inversely proportional to the depth. The magnitude of $U$ can be crudely estimated from (6) as $\tau/(\rho H \omega)$, compared with the magnitude of $u'$ estimated, again crudely, from (17) as $\tau/\rho (\mu \omega)^k$. With these estimates, and the parameter values listed in Fig. 1, the current magnitudes are equal in 10 m of water, while the depth-varying current $u'$ is an order-of-magnitude stronger in 100 m. These values are indicative only; a more comprehensive discussion of the relationship between the two components of the flow may be found, for example, in CSANADY (1982).

### 4. FORCING AT THE INERTIAL FREQUENCY

As the forcing frequency $\omega$ approaches the inertial frequency $f$, the amplitude of the clockwise wave tends to infinity, according to (18), while its wavelength, decay depth scale and phase velocity also become infinite (Table 1). In reality, once the vertical decay scale exceeds the water depth, the water depth itself becomes the dominant vertical scale. In this case, formula (18), for the wave amplitude, is no longer valid because it should now also take account of the reflected wave amplitude $c_2$.

When $\omega = f$, the solutions (12) and (13) for $u'$ and $v'$ no longer hold. Now the two velocity components are given by

$$
\begin{align*}
    u' &= u^a + u^c \\
    v' &= iu^a - iu^c,
\end{align*}
$$

where

$$
\begin{align*}
    u^a &= (a_1 e^{a_1 z} + a_2 e^{-a_2 z + H}) e^{-i\omega t} - \frac{1}{4fH} \left[ i \left( \frac{\tau^x}{\rho} - ru_b \right) + \frac{\tau^y}{\rho} - rv_b \right] \\
    u^c &= \frac{1}{4 \mu H} \left[ \frac{\tau^x}{\rho} - ru_b + i \left( \frac{\tau^y}{\rho} - rv_b \right) \right] \left( z^2 - \frac{H^2}{3} \right) + b \left( z + \frac{H}{2} \right) e^{-i\omega t}
\end{align*}
$$

As before, the terms in $u^a$ in (20) rotate anticlockwise in time, driven by the anticlockwise component of the wind, while the terms in $u^c$ rotate clockwise. The constant terms in (21) again occur to ensure that the depth integrals of $u'$ and $v'$ are zero. The constants $a_1$ and $a_2$ are the amplitudes of the top- and bottom-generated, anticlockwise waves, as in (12) and (13), where $a_1$ has the value given by (14). The unknown amplitude $b$ in the clockwise solution now multiplies a linear, rather than a wavelike, function in $z$. The full clockwise solution is quadratic in $z$. The bottom velocities $u_b$ and $v_b$ in (21) are also unknown constants.

Only the clockwise solution will be considered in this section, since the anticlockwise waves do not exhibit exceptional behaviour at $\omega = f$. The clockwise solution can again be
isolated by specifying a clockwise wind stress, that is

$$\tau^x = \tau^c \tau^y = -i \tau^c.$$ 

With this wind stress, \(u^e\) in (20) and (21) is zero.

Because the depth scale is now the water depth, it is not possible to ignore, as was done in the previous section, the bottom velocities \(u_b\) and \(v_b\). However, we will ignore the contribution of the depth-averaged velocities \(U\) and \(V\) to \(u_b\) and \(v_b\). This is done principally for convenience, since the global solution is not being considered in the present paper. However, as discussed at the end of Section 3, neglect of \(U\) and \(V\) at the bottom may often prove to be a good assumption.

There are three unknowns in \(u^c\), that is \(b\), \(u_b\) and \(v_b\). The unknowns are determined from the surface and bottom boundary conditions (4) and (5), and the equation

$$u(z = -H) = u_b.$$ 

The solution for \(u^c\) is then

$$u^c = \frac{\tau^c}{\rho \mu} \left\{ \frac{3}{2H} \frac{2\mu + rH}{3\mu + rH} \left( \frac{z^2 - H^2}{3} \right) + z + \frac{H}{2} \right\} e^{-i\omega t}. \tag{22}$$

An immediately obvious consequence of (22) is that, at the inertial frequency, there is no spiralling of the velocity vector with depth. The velocity vector is parallel to the wind stress, and rotates with the wind stress, at all depths through the water column.

From the surface boundary condition (4), it may be seen that, if the depth scale is the water depth, the velocity vector should scale as the wind stress multiplied by the depth. That is, as the depth increases, the magnitude of the velocity response to a given wind stress increases proportionally. In (22), as the depth increases so that \(rH \gg \mu\),

$$u^c = \frac{\tau^c}{4\rho \mu H} (z + H) (3z + H) e^{-i\omega t}. \tag{23}$$

Thus, in deep water, the velocity is zero at one-third of the water depth, and at the bottom. In the lower two-thirds of the water column, the currents are in the opposite direction to the wind, with the maximum counter-flowing current at two-thirds of the water depth.

The profile (22) is plotted in Fig. 2 for water depths of 50, 100 and 200 m. For these depths, and the assumed values of \(r\) and \(\mu\), the ratio \(rH/\mu\) has the values 1.5, 3 and 6, respectively. The deep-water approximation (23) is obviously increasingly valid in these cases.

5. FREE WAVES AND THE TRANSIENT SOLUTION

The inertial solution occurs when the wind forcing has the same frequency and is in the same rotational direction as the free waves of the ocean system. In this section, we shall examine the nature of the free waves, and determine how such waves are generated and decay when the system is started up from rest.

Free waves are the oscillatory solutions to (9) and (10) when \(\tau^x\) and \(\tau^y\) are both zero. They are best examined by introducing the complex velocity

$$we^{-i\omega t} = u + iv, \tag{24}$$
where $\omega$ is now the free-wave frequency, which is to be determined. Using (9) and (10) the equation for $w$ is now

$$-i(\omega - f)w = \mu w_{zz} + \frac{r}{H} w_b,$$

(25)

where $w_b$ is the value of $w$ evaluated at $z = -H$. The surface and bottom boundary conditions follow directly from (4) and (5). The solution to (25) may be obtained by observing that, if

$$\omega = f - ik^2 \mu,$$

(26)

with $k$ a real constant, then (25) reduces to a real eigenvalue problem

$$w_{zz} + k^2 w = -\frac{r}{\mu H} w_b.$$

(27)

The solution to (27) is then

$$w = A \left( \cos k z - \frac{\sin k H}{k H} \right),$$

(28)

where $A$ is an arbitrary amplitude, and $k$ satisfies the transcendental equation

$$\tan k H = \frac{r k H}{r + \mu k^2 H}.$$

(29)

There is a countable infinity of discrete positive eigenvalues $k$, given by the solutions to (29), which we will denote, in ascending order $k_1, k_2, \ldots$. The corresponding eigen-
functions, given by (28), and denoted by \(w_1, w_2, \ldots\), are the free waves of the system. The eigenfunctions are even functions of \(k\), so that the negative eigenvalues, \(-k_1, -k_2, \ldots\), do not give independent solutions. Further, the zero solution to (29) yields only the zero solution for \(w\).

We may note immediately that, by (26), these waves have time-dependence \(\exp(-i\omega t - k^2 \mu t)\); that is they are damped inertial waves. The e-folding decay time of the waves is \(1/k^2\mu\), which decreases with increasing \(k\). The higher the mode, the more rapid is its decay.

For the parameter values used in other examples in this study, and for a depth of 100 m, the first three eigenvalues are \(k_1 = 0.0373\), \(k_2 = 0.0668\), and \(k_3 = 0.0972\), with corresponding decay time-scales of 20, 6 and 3 h, respectively. The profiles of the first three eigenfunctions are shown in Fig. 3. As for the forced inertial waves, the velocity vectors for the free inertial waves are parallel at all depths and rotate clockwise in time.

The eigenfunctions \(w_1, w_2 \ldots\) are orthogonal. This may be shown in the standard manner by multiplying the equation (27) for a function \(w_j\) by \(w_i\), subtracting that from \(w_j\) times the equation for \(w_i\), and integrating over the depth using the boundary conditions. Using the orthogonality, it is straightforward to examine the manner in which the free waves are generated when a system is started up from rest with an oscillatory wind of the form considered in the previous sections. The free waves then decay, leaving the steady oscillatory solution.

As an example of such dynamics, we shall consider the establishment of the inertial solution (22) discussed in Section 4. The forced wave \(u^f\) in (22) can be expressed as a sum of the eigenfunctions at \(t = 0\) as

\[
u^f(t = 0) = \sum_{n \geq 1} a_n w_n(z),
\]

where the \(a_n\)'s are real constant amplitudes. The full solution for the system started from rest at \(t = 0\) is then given by

\[
u = u^f - \sum_{n \geq 1} a_n w_n(z) \exp(i\omega t - k^2 \mu t).
\]
Figure 4 shows the profile (31) over the first two inertial periods after starting from rest. The currents grow smoothly through the water column, and the steady oscillatory profile (Fig. 2b) is almost fully established after two inertial periods (35 h), consistent with the first mode decay scale of 20 h. Given the rapid decay of the higher modes, the profile is established quickly (relative to the inertial period) over the full depth of the water column. It should be recalled, in interpreting Fig. 4, that the profile rotates with the inertial period, and that there is no instantaneous motion normal to the plane of the profile.

CRAIG (1988) presents examples of the establishment of oscillatory profiles at frequencies other than the inertial (calculated, however, on the assumption of zero bottom friction). At non-inertial frequencies the velocity profile is not planar and, depending on the vertical decay scales, defined in Table 1, need not penetrate to the bottom. The solution of Fig. 4 is, however, indicative of the transitory response to non-oscillatory forcing such as, in this case, the impulsive initiation of the wind stress.

6. DISCUSSION

Interpretation in a non-rotating reference frame

We have seen that, to understand the current structure forced by periodic winds, the vertical profile should be interpreted as a pair of counter-rotating components. The motion is fundamentally rotational because it is taking place in a rotating frame of reference, the $f$-plane. In this section, we shall show how the dynamics of each of the rotational components is, in fact, more easily understood in a fixed reference plane.

We return to motion having an imposed frequency $\omega$, but, for economy of notation, $\omega$ is now allowed to take both positive and negative values. A wind stress, with amplitude $T$, having the form

$$\tau^x = Te^{-i\omega t}, \tau^y = -iTe^{-i\omega t},$$

(32)
and the associated velocity satisfying
\begin{equation}
\nu' = -iu', \tag{33}
\end{equation}
both rotate clockwise if \( \omega \) is positive, and anticlockwise for negative \( \omega \). The governing equations (9) and (10) can now be replaced by (33) and
\begin{equation}
-i(\omega - f) u' = \mu u_{zz} - \frac{Te^{-iot}}{\rho H} + \frac{r}{H} u_b. \tag{34}
\end{equation}

In the \( f \)-plane approximation, the frame of reference is rotating anticlockwise at an angular frequency of \( f \), relative to a non-rotating frame. A vector rotating with frequency \( \omega \) in the rotating frame then has frequency
\begin{equation}
\sigma = \omega - f \tag{35}
\end{equation}
in the fixed frame. The frequency has appeared frequently in the form (35) in previous sections (e.g. Table 1). In the fixed frame, retaining the same notation for the transformed velocity components, (34) becomes
\begin{equation}
-i\sigma u' = \mu u_{zz} - \frac{Te^{-i\sigma t}}{\rho H} + \frac{r}{H} u_b, \tag{36}
\end{equation}
in which all variables now have time dependence \( \exp(-i\sigma t) \). It is clear, from (36), that the motion can now be visualized, for simplicity of interpretation, in \( x \) and \( y \) coordinates. For consistency with previous sections, however, we shall continue to talk in terms of rotational vectors.

The solution to (36) is a rotating spiral, with e-folding depth scale
\begin{equation}
h = \left| \frac{2\mu}{\sigma} \right|^\frac{1}{2}, \tag{37}
\end{equation}
exactly as described in Sections 2 and 3. In the fixed frame, however, the solution is more easily visualized. There is no dependence in (36) on the inertial frequency, nor on the direction of rotation. The current profile rotates with the wind, with the phase lag of the current behind the wind increasing with depth. The higher the frequency of rotation, the stronger are the vertical shears. Thus, the vertical wavelength, which is the depth over which a full rotation of the current vector occurs, decreases as \( \sigma \) increases. Further, with the higher shear, the vertical decay scale decreases with increasing \( \sigma \).

It should be noted that, in the fixed frame, the two counter-rotating waves described in Sections 2 and 3 have frequencies that are different in magnitude. This is the reason that the properties of the two waves, including the depth scale (Table 1), are different.

If \( \omega \) is zero, \( \sigma = -f \), corresponding to Ekman's (1905) "steady" solution. In the fixed reference frame, this does not appear as a special case. The depth \( h \) in (34) is then the Ekman depth. There is only one depth scale in this case, rather than the two in (14), because the wind stress can be represented by a single, rather than a pair of, rotating vectors.

If \( \omega = f \), the inertial frequency, \( \sigma \), is zero. That is, the observed inertial wave is not a wave at all, but is time-independent in a fixed frame. In this case, the solution to (33) is quadratic in \( z \), as described in Section 4. Similarly, in the fixed frame, the "free-wave"
solutions of Section 5 are not wavelike in time, but are wavelike in the vertical and decay in time, with time-constant $k^2\mu$, where $k$ is given by (29).

7. CONCLUSIONS

Implications for data analysis

The techniques and solutions described throughout this paper are intended principally as a tool for understanding data either collected in the field or simulated in a modelling exercise. The approach that has been advocated is basically the rotary spectral method described by Gonella (1972). This is effectively the approach adopted in the tidal (Tee, 1979; Prandle, 1982a, b; Soulsby, 1983; Maas and van Haren, 1987) and wind-driven (Webster, 1986) studies referred to in Section 1.

Separation of the local dynamics, with which this paper has been concerned, from the global dynamics requires simple depth-integration of the currents which, in turn, requires a reasonable degree of vertical resolution in a field experiment. Selection of the frequencies of interest is then done either by spectral or harmonic analysis. If rotary spectral analysis is used, the spectral components are already in the form of counter-rotating vectors. Otherwise these vectors are simply formed by the addition of out-of-phase components, as indicated by (33).

The major conclusion of this present study is that the rotational components are best examined in the non-rotating reference frame, achieved by the frequency translation (35). If the eddy viscosity is constant, the solutions described in the previous sections will be directly applicable. If the eddy viscosity is not approximately constant, then the concepts, summarized in the previous section, will still be valid. In a field study, it would in fact be viable to use the equations in an inverse manner, to determine the vertical eddy viscosity distribution.

The global solution has been ignored throughout this study. By boundary condition (5), the global solution acts on the local solution through a bottom stress analogously to the wind stress at the surface. Thus, in a data set, the influence of the global dynamics on the local solution may be interpreted in the same rotational setting as we have used for the wind stress.

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